

Minimal sets of fibre-preserving maps in graph bundles

S. Kolyada · L. Snoha · S. Trofimchuk

Abstract Topological structure of minimal sets is studied for a dynamical system (E, F) given by a fibre-preserving, in general non-invertible, continuous selfmap F of a graph bundle E . These systems include, as a very particular case, quasiperiodically forced circle homeomorphisms. Let M be a minimal set of F with full projection onto the base space B of the bundle. We show that M is nowhere dense or has nonempty interior depending on whether the set of so called endpoints of M is dense in M or is empty. If M is nowhere dense, we prove that either a typical fibre of M is a Cantor set, or there is a positive integer N such that a typical fibre of M has cardinality N . If M has nonempty interior we prove that there is a positive integer m such that a typical fibre of M , in fact even each fibre of M over a *dense open* set $\mathcal{O} \subseteq B$, is a disjoint union of m circles. Moreover, we show that each of the fibres of M over $B \setminus \mathcal{O}$ is a union of circles properly containing a disjoint union of m circles. Surprisingly, some of the circles in such “non-typical” fibres of M may intersect. We also give sufficient conditions for M to be a sub-bundle of E .

Keywords Dynamical system · minimal set · graph bundle · skew product

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1 Introduction and statement of main results

Let f be a continuous selfmap of a compact metric space X . The system (X, f) is called *minimal* if there is no proper subset $M \subseteq X$ which is nonempty, closed and f -invariant (i.e., $f(M) \subseteq M$). In such a case we also say that the map f itself is minimal. Clearly, a system (X, f) is minimal if and only if the orbit $\{x, f(x), f^2(x), \dots\}$ of every point $x \in X$ is dense in X . Note that by an orbit we mean a forward orbit rather than a full orbit, even if f is a homeomorphism. The basic fact is that any compact dynamical system (X, f) has minimal (closed) subsystems $(M, f|_M)$. Such closed sets M are called *minimal sets* of f or, more precisely, of (X, f) . The minimal sets, as ‘irreducible’ parts of a system, attract much attention and their topological structure is one of the central topics in topological dynamics.

The classification of compact metric spaces admitting minimal maps is a well-known open problem in topological dynamics [2, 10]. For the state of the art of the problem see [3, 8, 9, 22] and references therein.

It is folklore that if X is a compact *zero-dimensional* space, $f : X \rightarrow X$ is continuous and $M \subseteq X$ is a minimal set of f then M is either a finite set (a periodic orbit of f) or a Cantor set and this is in fact a characterization because also conversely, whenever $M \subseteq X$ is a finite or a Cantor set then there is a continuous map $f : X \rightarrow X$ such that M is a minimal set of f . Among *one-dimensional* spaces, the characterization of minimal sets is known for *graphs* — minimal sets on graphs are finite sets, Cantor sets and unions of finitely many pairwise disjoint simple closed curves, see [4] or [29]. The full characterization of minimal sets on (local) *dendrites* has been found just recently [3].

In *higher dimensions* the topological structure of minimal sets is much more complicated and, besides some important examples (see e.g. [9, 18]), only few results are known. One obvious fact is that if h is a homeomorphism of a connected space X then a minimal set of h either is nowhere dense or coincides with X . It is interesting that the same conclusion is true for continuous endomorphisms of compact connected 2-manifolds [25] while it is an open problem whether this result holds also in dimensions $n > 2$. A related question is which manifolds admit minimal maps. Again, the answer is completely known only in dimension 2: among 2-manifolds, compact or not, with or without boundary, only finite unions of tori and finite unions of Klein bottles admit minimal maps [8]. In dimensions higher than 2 the tori and we know from [13] that also the odd-dimensional spheres admit minimal diffeomorphisms. Note that a non-compact manifold never admits a minimal map by [16]. This is because we define minimality as density of *forward* orbits. It does not exclude the possibility to have a homeomorphism of a non-compact manifold with all *full* orbits dense. In any case, 2-sphere without a finite set of points does not admit such a homeomorphism [27].

To find a full topological characterization of minimal sets on compact, connected 2-manifolds is a very difficult task. Very recently, a classification of minimal sets on 2-torus has been obtained for homeomorphisms [22].

The main contribution of the present paper is a partial description of minimal sets of fibre-preserving maps in graph bundles.

1.1 Fibre-preserving maps and their minimal sets

A dynamical system (E, F) is called an *extension* of a *base* dynamical system (B, f) if there is a continuous surjective map $p : E \rightarrow B$, called a *factor map* or a *projection*, such that $p \circ F = f \circ p$. We also say that the base (B, f) is a *factor* of (E, F) . Note that for every $b \in B$ we have $F(p^{-1}(b)) \subseteq p^{-1}(f(b))$, i.e., F sends the fibre over b into the fibre over $f(b)$. Therefore F is said to be *fibre-preserving*. Suppose that (B, f) is minimal and (E, F) is an extension of it. If we additionally assume that E is compact then always there is a minimal set M in the system (E, F) and since M projects onto a minimal set of (B, f) , we necessarily have $p(M) = B$.

A very special case of an extension is when E is a cartesian product, $E = B \times Y$, and $F(x, y) = (f(x), g(x, y))$. Then F is fibre-preserving, the fibres being the “vertical” copies of Y , i.e. the sets $\{b\} \times Y$ where $b \in B$, and the factor map being the natural projection of E onto B . The map F is also called a *skew product map* or sometimes a *triangular map*.

The study of fibre preserving maps and their minimal sets has a long tradition. Much attention has been paid to minimal sets of fibre-preserving maps on the torus, for instance in the case of *quasiperiodically forced (qpf) circle homeomorphisms*. These systems naturally appear in the study of the scalar linear quasi-periodic Schrödinger equations. In such a case the dynamics is given by the *projective action of a quasiperiodic $SL(2, \mathbb{R})$ -cocycle* (the 2-torus is identified with $\mathbb{S}^1 \times \mathbb{P}^1(\mathbb{R})$ and the projective action of $SL(2, \mathbb{R})$ is considered on $\mathbb{P}^1(\mathbb{R})$). The most interesting situation occurs when the mentioned Schrödinger equations are non-uniformly hyperbolic [7]. An old question by Herman [19, Section 4.14] concerns topological structure of the unique minimal set M in that case. Herman partially described the set M . In particular, M is nowhere dense and the intersection M_θ of M with a vertical fibre $\{\theta\} \times \mathbb{P}^1(\mathbb{R})$ is, generically, a singleton. Herman’s question is whether also all the other fibres M_θ are connected; for more details and related results see [5], [6], [7], [19] and references therein. Bjerklov [7] shows that the question has an affirmative answer in some special cases. According to recent preprint by Hric and Jäger [20], in general the answer is negative.

In a more general setting of *skew product circle flows* (both continuous and discrete) over a minimal base (forcing) on a compact metric space Y , a topological classification of minimal sets was recently given by Huang and Yi [21]. They showed that if M is a minimal set of such a system then either M is the whole space $Y \times \mathbb{S}^1$, or there is a positive integer N such that a typical fibre of M consists of N points, or a typical fibre of M is a Cantor set. Below in Theorem E, we amplify this result to general fibre-preserving (not necessarily invertible) maps in compact graph bundles over a minimal base.

Béguin, Crovisier, Jäger and Le Roux [5] have constructed transitive qpf circle homeomorphisms with complicated minimal sets. For example, the minimal set can be a Cantor set whose intersection with each vertical fibre (circle) is uncountable (the possibility that some of these intersections have isolated points in the topology of the fibre has not been excluded and is probable). Thus, minimal sets of fibre preserving maps can be quite complicated. This is true even for triangular maps in the square. To illustrate this, recall that so called *Floyd-Auslander minimal systems* [17]

are homeomorphisms which are extensions of Cantor minimal homeomorphisms and their phase spaces are subsets of the unit square which are nonhomogeneous — some fibres are compact intervals while the others are singletons. Modifying the construction, one can obtain also a noninvertible nonhomogeneous system of this kind [33]. Note that, by the extension lemma from [23], all these systems can be embedded into systems given by triangular selfmaps of the square.

In the present paper we wish to shed more light on the problem of characterizing minimal sets of higher dimensional maps by studying minimal sets of continuous fibre-preserving (not necessarily invertible) maps in graph bundles. It does not seem easy to generalize the results to more general bundles.

1.2 Star-like interior points and end-points in graph bundles

To state our main results, we need some terminology. A *fibre space* is an object (E, B, p) where E and B are topological spaces and $p : E \rightarrow B$ is a continuous surjection. Here E , B and p are called the total space, the base (space) and the projection (map) of the fibre space, respectively, and $p^{-1}(b)$ is called the fibre over the point $b \in B$. If Γ is another topological space, the fibre space (E, B, p) is called a *fibre bundle* with fibre Γ , and denoted by (E, B, p, Γ) , if the projection map $p : E \rightarrow B$ satisfies the following condition of local triviality: For every point $b \in B$ there is an open neighborhood U of b (which will be called a *trivializing neighborhood*) and a homeomorphism $h : p^{-1}(U) \rightarrow U \times \Gamma$ such that on $p^{-1}(U)$ it holds $\text{pr}_1 \circ h = p$. Here $\text{pr}_1 : U \times \Gamma \rightarrow U$ is the canonical projection onto the first factor. We will always assume that **both E and B are compact metric spaces** and so we will speak on compact fibre bundles.

Given a fibre space (E, B, p) , consider dynamical systems (E, F) and (B, f) with $p \circ F = f \circ p$. Thus, (E, F) is an extension of (B, f) and (B, f) is a factor of (E, F) , the projection map p being the factor map. Then F is *fibre-preserving*, it sends the fibre $p^{-1}(b)$ over $b \in B$ into the fibre $p^{-1}(f(b))$ over $f(b)$.

A *graph* is a (nonempty) compact metric space which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end-points. A graph need not be connected and a singleton is not a graph. A *tree* is a graph containing no *circle* (i.e. a simple closed curve). The number of arcs emanating from a point $x \in G$ is called the *order* of x and is denoted by $\text{ord}(x, G)$. Points of order 1 are called *end-points* of G and points of order at least 3 are called *ramification points* of G .

For $n \geq 1$ we will consider the notion of the *n-star* S_n , which can be described as the set of all complex numbers z such that z^n is in the real unit interval $[0, 1]$, i.e., a central point (the origin) with n copies of the interval $[0, 1]$ attached to it. We will view the *n-star* as a tree with $n + 1$ vertices, one of them (the central point) having order n and the other n vertices (the end-points of S_n) having order 1. Any set homeomorphic to S_n will also be called an *n-star* and also denoted by S_n . Note that both S_1 and S_2 are homeomorphic to a closed interval. By the *open n-star* Σ_n we will mean S_n without its n end-points. In particular, Σ_2 is homeomorphic to an open interval (while Σ_1 to a half-closed interval).

Definition 1 Let Γ be a graph and $Z \subseteq \Gamma$ be closed. A point $x \in Z$ is said to be a *star-like interior point* of Z if there exists a Z -open neighborhood of x (i.e., the intersection of Z and a Γ -open neighborhood of x) which is homeomorphic to Σ_k for some $k \geq 2$; we assume here that this homeomorphism sends the point x to the central point of Σ_k (then k is uniquely determined). If $x \in Z$ is not a star-like interior point of Z we say that it is an *end-point* of Z . Let $\text{Sint}(Z)$ and $\text{End}(Z)$ denote the set of all star-like interior points of Z and the set of all end points of Z , respectively.

Figure 1 shows that a star-like interior point of Z need not be an interior point of Z in Γ and an interior point of Z need not be a star-like interior point of Z .

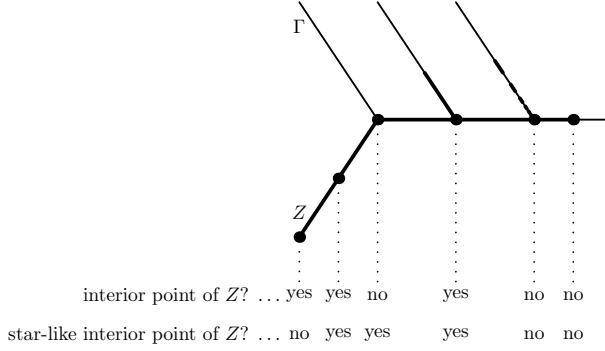


Fig. 1 There is no connection between interior and star-like interior points.

The set $\text{Sint}(Z)$ is open in Z (but not necessarily in Γ) and so the set $\text{End}(Z)$ is closed in Z (hence closed in Γ). If Z is a subgraph of Γ , the set $\text{End}(Z)$ coincides with the usual set of end-points of the graph Z .

A *graph bundle* is a fibre bundle whose fibre Γ is a graph. Given a graph bundle (E, B, p, Γ) , for $M \subseteq E$ and $b \in B$ we denote $M_b = M \cap p^{-1}(b)$; this set is said to be the *fibre of M over b* . When speaking on the fibres of M over points lying in a subset U of B , we sometimes call them *fibres of M over the set U* . If $M \subseteq E$ and $U \subseteq B$, we denote $M_U = M \cap p^{-1}(U)$. So, M_U is the union of all fibres of M over the set U .

Definition 2 Given a closed set M in a compact graph bundle (E, B, p, Γ) we define the set of *star-like interior points of M* and the set of *end-points of M* by, respectively,

$$\text{Sint}(M) = \bigcup_{b \in B} \text{Sint}(M_b) \quad \text{and} \quad \text{End}(M) = \bigcup_{b \in B} \text{End}(M_b).$$

Of course, $\text{End}(M) = M \setminus \text{Sint}(M)$. In general it is not true that $\text{Sint}(M)$ is open or $\text{End}(M)$ is closed in E or M .

1.3 Main results

Throughout the paper, (E, B, p, Γ) is a compact graph bundle, (E, F) and (B, f) are dynamical systems with $p \circ F = f \circ p$. We also assume that the base system (B, f) is minimal or, equivalently, that $p(M) = B$ for each minimal set $M \subseteq E$ of F . Our first

main result is the following dichotomy for a minimal set M formulated in terms of end-points of M .

Theorem A. *Let M be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle (E, B, p, Γ) . Then there are two mutually exclusive possibilities:*

- (A1) $\overline{\text{End}(M)} = M$ (and this holds if and only if M is nowhere dense in E);
- (A2) $\text{End}(M) = \emptyset$ (and this holds if and only if M has nonempty interior in E).

In particular, the fibre-preserving maps in tree bundles have only nowhere dense minimal sets.

The assumption that the base system (B, f) is minimal is not restrictive. In fact, if M is a minimal set of (E, F) then its projection $p(M)$ is a minimal set of (B, f) and so one can pass to the sub-bundle over $p(M)$ and to consider, instead of (E, F) , the system $(E^*, F|_{E^*})$ where $E^* = p^{-1}(p(M))$. As an application of this fact we get that though a minimal set of a triangular map in the square can contain a vertical interval (so that in general $\text{End}(M) \neq M$ in the case (A1)), the following corollary holds (I denotes a real compact interval and pr_1 is the projection onto the first coordinate).

Corollary B. *Let $F(x, y) = (f(x), g(x, y))$ be a continuous triangular map in the square I^2 and let M be a minimal set of F . Then M is nowhere dense in the space $\text{pr}_1(M) \times I$.*

We know from the characterization of minimal sets on the interval that $\text{pr}_1(M)$ is either a finite set or a Cantor set. In the latter case the result in the corollary is nontrivial, it strengthens Theorem 1 from [14] (where the same result is obtained for a very particular and small subclass of the class of triangular selfmaps of the square) and answers in negative the question posed by J. Smítal whether a minimal set M of a triangular map in the square can have nonempty interior in the space $\text{pr}_1(M) \times I$.

So, no direct-product $B \times I$ admits a minimal fibre-preserving map (with the fibre I). Cannot we remove the assumption that the maps are fibre-preserving? The answer is negative. In fact, if \mathbb{S}^1 is a circle and H is the Hilbert cube then the space $P = \mathbb{S}^1 \times H$ admits a continuous minimal map (in the form of a skew product map with an irrational rotation in the base \mathbb{S}^1 and homeomorphisms $H \rightarrow H$ as fibre maps, see [15]). However, P can be written in the form $P = (\mathbb{S}^1 \times H) \times I$. Thus we have a space of the form $B \times I$ admitting a minimal, of course not fibre-preserving map (with the fibre being I). Here dimension of B is infinite. An interesting question is whether it is true that all minimal, not necessarily fibre-preserving, maps in interval bundles $B \times I$ have only nowhere dense minimal sets if we additionally assume that B has finite dimension. Recall that, by the result from [25], this is true if B is a one-dimensional manifold, possible with boundary, so that $B \times I$ is a 2-manifold with boundary.

In each of the cases (A1) and (A2) in Theorem A, there are severe restrictions for the topological structure of the minimal set M . In the case (A2), some of such restrictions are listed in Theorem C whose *full version* is given in Section 6. Here, in Introduction, we prefer to list just those of them which seem to be most important and whose statement is neither cumbersome nor involves the notion of *strongly* star-like

interior points which will be introduced in Section 4. To keep the shortened version of the theorem compatible with the full version, we do not renumber the items.

Theorem C (shortened version). *Let M be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle (E, B, p, Γ) . Assume that M has nonempty interior. Then the following holds.*

- (C4) *All the sets M_b , $b \in B$, are unions of circles. In fact there exist an open dense set $\mathcal{O} \subseteq B$ and a positive integer m such that*
 - *for each $z \in \mathcal{O}$, M_z is a disjoint union of m circles, and*
 - *for each $z \in B \setminus \mathcal{O}$, M_z is a union of circles which properly contains a disjoint union of m circles.**In particular, if M_z is a circle for some $z \in B$, then M_z is a circle for all z in the open dense subset \mathcal{O} of B .*
- (C6) *The set $M_{\mathcal{O}}$ is dense in M .*
- (C8) *If $z \in \mathcal{O}$ then the set M_z , which is a disjoint union of m circles, is mapped by F onto a disjoint union of m circles in $M_{f(z)}$.*
- (C10) *If f is monotone then $\mathcal{O} = B$ (hence, M is a sub-bundle of E).*
- (C11) *If $E = B \times \Gamma$ and B is locally connected then $\mathcal{O} = B$ (hence, M is a sub-bundle of E and if B is also connected, then M is a direct product).*

Properties of the map $F|_M$ are partially described in Proposition 2. The next result shows that $\mathcal{O} \neq B$ is possible and that some circles in a fibre of M over a point in $B \setminus \mathcal{O}$ can intersect.

Theorem D. *There is a minimal selfmap f of a Cantor set B , a connected graph Γ and an extension $(B \times \Gamma, F)$ of (B, f) with a minimal set M such that, for some $b \in B$,*

- *M_z is a circle for each $z \neq b$, and*
- *M_b is a union of two circles. Depending on the choice of such a system, the union of any two different circles in any graph can appear as the set M_b .*

Recall that a set in a Baire space is called *residual* if its complement is of 1st category, i.e. a countable union of nowhere dense sets. By saying that a *typical* (or *generic*) fibre of M has some property we mean that there is a residual set in the base B such that for each b in this residual set, the fibre M_b of M has this property.

Notice that Theorem C, part (C4), describes a typical fibre of the minimal set M in the case (A2). Also in the case (A1) we are able to describe a typical fibre of M .

Theorem E. *Let M be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle (E, B, p, Γ) . Assume that M is nowhere dense. Then either*

- (E1) *a typical fibre of M is a Cantor set, or*
- (E2) *there is a positive integer N such that a typical fibre of M has cardinality N .*

The number N in (E2) is given by the formula from Proposition 3 in Section 7. Even if F is a homeomorphism, one cannot claim that all fibres of M have the same cardinality, see examples in the next section.

In the special case when E is a direct product $B \times \Gamma$, Γ is the circle and $F : E \rightarrow E$ is a homeomorphism, Theorem E has been known from [21, Theorem 6.1].

Notice the following asymmetry: in the case (A2) we know from (C4) that a “non-typical” fibre of M is a union of circles, while in the case (A1) the topological structure of a “non-typical” fibre is unknown even for the qpf circle homeomorphisms and the triangular maps in the square (as Floyd-Auslander systems show, some of these fibres can contain nondegenerate intervals).

The paper is organized as follows. In Section 2 we present several illustrating examples of minimal sets of fibre-preserving maps in graph bundles and we also prove Theorem D. Section 3 contains some dynamical and topological preliminaries. Then, in Section 4 we introduce the key notion of our paper, namely that of a strongly star-like interior point of a subset of a graph bundle, and we study the structure of open neighborhoods of those compact subsets of a fibre which entirely consist of strongly star-like interior points of a given subset of the bundle. The proofs of Theorems A, C and E are given in Sections 5, 6 and 7, respectively.

2 Some examples and proof of Theorem D

Theorems C and E give necessary conditions for subsets of graph bundles to be minimal for a fibre-preserving map. Observe the following.

Suppose that the base B is a singleton and so E is just Γ . Then Theorems C and E imply that minimal sets on graphs are finite sets, Cantor sets or disjoint unions of (finitely many) circles. This is already a characterization of minimal sets on graphs, as shown in [4] or [29]. If B is finite (and so the minimal base system is just a periodic orbit) we get that each fibre of M either is a Cantor set or consists of the same finite number of points or the same finite number of disjoint circles. Again, one can easily show that this is a characterization of minimal sets (with full projection) of fibre-preserving maps in graph bundles with finite base.

However, we do not know how far we are from a topological characterization of minimal sets (with full projection) of fibre-preserving maps in graph bundles with infinite base. Indeed, if typical fibres of some compact set $M \subseteq E$ are as described in Theorems C and E (and M has no isolated point, which would be an obvious obstacle for M to be minimal) then it is not easy to check when there exists a fibre-preserving map F in E such that M is a minimal set of F .

2.1 Examples of nowhere dense minimal sets

Only nowhere dense minimal sets can appear if Γ is a tree. Say, a triangular map in the square can have a minimal set which is the direct product of a Cantor set with itself. More interesting are the following examples of nowhere dense minimal sets which are not totally disconnected.

Example 1 (Floyd-Auslander minimal sets). By the extension lemma from [23] one can extend any Floyd-Auslander minimal system (M, H) (see [17]) to a triangular map defined on the product of the Cantor set (the projection of M) and a compact

interval. Though in this example H is a homeomorphism on M , it is not true in general that if f is a homeomorphism then $F|_M$ is monotone — to see it, replace (M, H) in this construction by a noninvertible modification of it from [33]. Other examples can be obtained in a similar way, by replacing a Floyd-Auslander minimal system by some other cantoroids (for the definition of a cantoroid see [3]). \square

Example 2 (Boundary of the Möbius band as a minimal set). Imagine, in \mathbb{R}^3 , a circle \mathbb{S}^1 in a horizontal plane and a vertical straight line segment I whose center is a point of \mathbb{S}^1 and the length of I is smaller than the radius of \mathbb{S}^1 . By moving I periodically along \mathbb{S}^1 in such a way that the center of I is always in \mathbb{S}^1 and during one period, when the center of I comes back to its initial position, we turn I upside down to obtain the Möbius band E . Here E is an interval bundle, \mathbb{S}^1 being the base space and the positions of I being the fibres over points of \mathbb{S}^1 . The described movement, when considering time from $-\infty$ to $+\infty$, gives a flow on E and each time- t map of this flow is a fibre-preserving map on E .

We can move I in such a way that for the time-1 map F of the mentioned flow, the restriction $f = F|_{\mathbb{S}^1}$ is an irrational rotation, by some angle α , of \mathbb{S}^1 . Hence \mathbb{S}^1 is a minimal set of F . Then the boundary ∂E of E is also a minimal set of F , since the restriction of F to ∂E is conjugate to $\alpha/2$ rotation of the circle.

Notice that the simple closed curve ∂E is a sub-bundle of E (the fibre having cardinality 2) but it is not a direct product of the base space \mathbb{S}^1 with a two-point set. \square

Example 3 (Sturmian minimal sets). Consider a Sturmian minimal system (\mathfrak{S}, σ) , see e.g. [34, pp. 200–203], satisfying the following properties: it is a minimal subshift of $\{0, 1\}^{\mathbb{Z}}$ and it is an almost one-to-one extension of a system $(\mathbb{S}^1; \text{rot}_\alpha)$, where \mathbb{S}^1 is the circle and rot_α is an irrational rotation. More precisely, if $\Sigma : (\mathfrak{S}, \sigma) \rightarrow (\mathbb{S}^1; \text{rot}_\alpha)$ is the corresponding factor map, then there is a countable dense set $D \subset \mathbb{S}^1$ such that for all $z \in \mathbb{S}^1 \setminus D$ the fibre $\Sigma^{-1}(z)$ consists of just one point of \mathfrak{S} and for all $z \in D$ the fibre $\Sigma^{-1}(z)$ consists of two points of \mathfrak{S} . We may think of \mathfrak{S} as being a minimal set of a fibre-preserving map in $\mathbb{S}^1 \times [0, 1]$, whose base map is rot_α . Let us explain this.

The point inverses of Σ are the fibres of the mentioned almost 1-to-1 extension and the homeomorphism σ sends fibres to fibres. Topologically, \mathfrak{S} is a Cantor set (since the Sturmian system is an uncountable minimal subshift) and so we may assume that $\mathfrak{S} \subseteq [0, 1]$. Consider the map $H : \mathfrak{S} \rightarrow \mathbb{S}^1 \times [0, 1]$ sending $s \in \mathfrak{S}$ to $(\Sigma(s), s) \in \mathbb{S}^1 \times [0, 1]$. Then H is continuous and injective, so it is an embedding of the set \mathfrak{S} into the cylinder $\mathbb{S}^1 \times [0, 1]$. Moreover, vertical fibres of the Cantor set $H(\mathfrak{S}) \subseteq \mathbb{S}^1 \times [0, 1]$ correspond to point inverses of Σ which means that H induces fibre-preserving dynamics on $H(\mathfrak{S})$ which is topologically conjugate to σ .

Again, by the extension lemma from [23], one can extend this dynamics on $H(\mathfrak{S})$ to a fibre-preserving map $F : \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{S}^1 \times [0, 1]$. Then $H(\mathfrak{S})$ is a minimal set of F having singleton fibres with the exception of countably many fibres, each of them consisting of two points. \square

2.2 Examples of minimal sets with nonempty interior

This case can occur only if the graph Γ contains a circle. As an example, consider an irrational rotation of the torus (M is the whole torus). To produce some more general “direct product” examples with B being a general compact metric space admitting a minimal map, one can use Proposition 1 and Corollary 2 below.

To prove Proposition 1, let us start by recalling a theorem due to H. Weyl (see e.g. [26, Chapter I, Theorem 4.1]) saying that if $(a_n)_{n=1}^\infty$ is a sequence of distinct integers then for almost all (with respect to the Lebesgue measure) real numbers x the sequence $(a_n x)_{n=1}^\infty$ is uniformly distributed modulo 1. As an obvious consequence of this theorem we get that for any sequence of positive integers $n_1 < n_2 < \dots$ there is an angle α such that the rotation g of \mathbb{S}^1 by the angle α is *minimal with respect to the sequence* $(n_k)_{k=1}^\infty$. This means that for every $s \in \mathbb{S}^1$ the set $\{g^{n_k}(s) : k = 1, 2, \dots\}$ is dense in \mathbb{S}^1 . Of course, any such rotation g is necessarily irrational.

The following simple proposition dealing with direct product maps (rather than with skew product minimal systems as for instance in [15]) is, though not most general possible, sufficient for our purposes. We present here a short proof, based on the Weyl’s theorem mentioned above.

Proposition 1 *Let (B, f) be a minimal dynamical system, B being a metric space. Then there exists an irrational rotation g of the circle \mathbb{S}^1 such that the direct product system $(B \times \mathbb{S}^1, f \times g)$ is minimal.*

Proof Fix $x_0 \in B$ and positive integers $n_1 < n_2 < \dots$ such that $f^{n_k}(x_0) \rightarrow x_0$ when $k \rightarrow \infty$. By the Weyl’s theorem, there is an irrational rotation g of \mathbb{S}^1 such that for every $s \in \mathbb{S}^1$ the set $\{g^{n_k}(s) : k = 1, 2, \dots\}$ is dense in \mathbb{S}^1 . We claim that $F = f \times g$ is minimal. It is sufficient to prove that the ω -limit set $\omega_F(x, s) = B \times \mathbb{S}^1$ for every $(x, s) \in B \times \mathbb{S}^1$.

From the choice of x_0 and g it follows that for every $y \in \mathbb{S}^1$, $\omega_F(x_0, y) \supseteq \{x_0\} \times \mathbb{S}^1$. Since the f -orbit of x_0 is dense in B and $F(\omega_F(x_0, y)) \subseteq \omega_F(x_0, y)$ and g is surjective, the closed set $\omega_F(x_0, y)$ contains the union of a dense family of fibres. We have thus proved that $\omega_F(x_0, y) = B \times \mathbb{S}^1$ for every $y \in \mathbb{S}^1$.

Now fix any point $(x, s) \in B \times \mathbb{S}^1$. Since $\omega_f(x) = B$ and \mathbb{S}^1 is compact, the set $\omega_F(x, s)$ contains at least one point $(x_0, y) \in \{x_0\} \times \mathbb{S}^1$. Then $\omega_F(x, s) \supseteq \omega_F(x_0, y) = B \times \mathbb{S}^1$. \square

Corollary 1 *Let $E = B \times \Gamma$ be a graph bundle such that B is a compact metric space admitting a minimal map and Γ be a graph containing a circle C . Then there exists a fibre-preserving map $F : E \rightarrow E$ such that $B \times C$ is a minimal set of F .*

Proof Using Proposition 1 extend a minimal map $f : B \rightarrow B$ to a minimal map $f \times g : B \times C \rightarrow B \times C$. Then use the fact that there is a retraction $r : \Gamma \rightarrow C$ and put $F = f \times (g \circ r)$. \square

However, for a general (i.e., not direct product) graph bundle (E, B, p, Γ) , where B is a compact metric space admitting a minimal map and Γ contains a circle, the existence of fibre-preserving maps having minimal sets with nonempty interior is not

clear at all. For instance, already the construction of such a minimal homeomorphism on the Klein bottle is not easy, see [11] or [32]. We do not know whether in any graph bundle which is not a tree bundle and whose base admits a minimal map there exists a fibre-preserving map having a minimal set with nonempty interior.

Recall that (X, f) is a *totally minimal* system if (X, f^n) is minimal for $n = 1, 2, \dots$.

Corollary 2 *Let (B, f) be a totally minimal dynamical system, B being a metric space. Let Γ be a graph which contains m disjoint circles. Denote the union of these circles by S . Then there exists a continuous map $h : \Gamma \rightarrow \Gamma$ such that $B \times S$ is a minimal set in the direct product system $(B \times \Gamma, f \times h)$.*

Proof Let g be the irrational rotation by angle α , which can be assigned to the minimal system (B, f^m) by Proposition 1. Fix a circle C in S . Let \tilde{g} be the map $S \rightarrow S$ whose restriction to C is conjugate to g and which is identity on $S \setminus C$. Then compose \tilde{g} with a homeomorphism on S , which cyclically permutes the m circles in S . Finally, extend the selfmap of S obtained in such a way to a continuous selfmap h of Γ (this is possible, see e.g. [4]). By Proposition 1, the set $B \times C$ is minimal for $(f \times h)^m = f^m \times h^m$ since $h^m|_C$ is conjugate to g . Then $B \times S$ is minimal for $f \times h$. \square

Example 4 (Torus attached to the boundary of the Möbius band as a minimal set). We construct a space E similarly as the Möbius band in Example 2 with only one difference – now, instead of moving the straight line segment I along the circle \mathbb{S}^1 , we move the graph Γ which is the segment I with two identical circles attached to I at the endpoints of I in such a way that the intersections of the circles with the straight line segment joining the centers of the circles are the endpoints of I . We assume that the diameter of Γ is smaller than that of \mathbb{S}^1 . So, E is a Möbius band whose boundary simple closed curve is replaced by a topological torus \mathbb{T}^2 .

As in Example 2, we consider the time-1 map F of the flow induced by the mentioned “movement” and put $f = F|_{\mathbb{S}^1}$, an irrational rotation of \mathbb{S}^1 by some angle α . The map F is fibre-preserving and we are going to extend it to a fibre-preserving continuous map $G : E \rightarrow E$ for which the torus \mathbb{T}^2 will be a minimal set.

Let $\varphi : \Gamma \rightarrow \Gamma$ be any continuous map such that the points of Γ which are symmetrical with respect to the center of I are mapped to symmetrical points (hence the center of I is a fixed point) and the restriction of φ to each of the two circles in Γ is an irrational rotation. The symmetry condition requires that both circles rotate by the same angle β and with the same “orientation”. Further, let $\Phi : E \rightarrow E$ be a continuous map which maps each of the fibres of E into itself in such a way that the restriction of Φ to each of the fibres is an isometric copy of φ (the fibres of E are isometric to Γ). Simply, in one of the fibres we choose an orientation of the circles (the same orientation), hence also the “orientation” of the β -rotations on them. The continuity of Φ then determines the “orientation” of the rotations on the circles in all other fibres. (Since we have the same orientation of the circles in Γ , one can see that this is a correct construction, we really get a well defined map Φ .)

Put $G = \Phi \circ F$. Then G is a fibre-preserving map on the graph-bundle E and the restriction of G to the torus \mathbb{T}^2 is a double rotation – irrational $\alpha/2$ -rotation in one direction and β -rotation in the other direction. Now we restrict ourselves to β for which G is a minimal map on \mathbb{T}^2 . Notice that, in contrast to Corollary 2, the obtained minimal set \mathbb{T}^2 is not a direct product of the base space \mathbb{S}^1 with a union of circles. \square

2.3 Proof of Theorem D

Given a set $A \subseteq \mathbb{R}^k$ and a vector $v \in \mathbb{R}^k$, by $A + v$ we mean the set $\{a + v : a \in A\}$.

Theorem D. *There are a minimal selfmap f of a Cantor set B , a connected graph Γ and an extension $(B \times \Gamma, F)$ of (B, f) with a minimal set M such that, for some $b \in B$,*

- M_z is a circle for each $z \neq b$, and
- M_b is a union of two circles. Depending on the choice of such a system, the union of any two different circles in any graph can appear as the set M_b .

Proof Case I: M_b is union of two disjoint circles.

Let (C, f) , with C being a subset of the real line, be a Cantor minimal system such that one point has two pre-images and all the other points have only one pre-image each. Such systems appear for instance in symbolic and interval dynamics. It will be convenient to give an explicit construction of such a system in order to introduce the notation which will be used throughout the whole proof. Start with the dyadic adding machine on the Cantor ternary set. Recall that it is often viewed as a restriction of an interval map to the invariant Cantor set, usually a restriction of the map shown for instance in [33, Fig. 1]; notice that then the adding machine is increasing at each point except at the rightmost one where it is decreasing. Choose a point a in this Cantor set which does not belong to the countable set consisting of the endpoints of the contiguous intervals (including the leftmost and the rightmost points of the Cantor set). Hence the points $a_{-j} := f^{-j}(a)$, $j = 1, 2, \dots$ do not belong to this countable set, too. Now blow up the backward orbit of a , i.e., for $j = 1, 2, \dots$, replace the point a_{-j} by a compact interval with length L_{-j} with convergent sum $\sum_{j=1}^{\infty} L_{-j}$ and remove the interior of this interval. This means that the points a_{-j} , $j = 1, 2, \dots$ are “doubled”, i.e. replaced by pairs of points $a_{-j}^- < a_{-j}^+$. What we get is again a Cantor set. Consider the dynamics on it which is inherited from the adding machine, except for the “new” points a_{-j}^-, a_{-j}^+ , $j = 1, 2, \dots$ where we still need to define the dynamics. To this end, send both a_{-1}^- and a_{-1}^+ to a and, since the adding machine is increasing at each a_{-j} and we want a continuous dynamics, for $j = 2, 3, \dots$ send a_{-j}^- to a_{-j+1}^- and a_{-j}^+ to a_{-j+1}^+ . The map defined in such a way is continuous and the system is minimal.

Recall that, up to a homeomorphism, there is only one Cantor set and it is homogeneous. Therefore, no matter which of the Cantor minimal systems (C, f) (such that one point has two pre-images and all the other points have only one pre-image) we choose, we may think of C as a Cantor set on the real line, with the point having two pre-images being for instance the rightmost point of C . For the same reason we can also assume that the two-preimages, denote them $c_l < c_r$, are the endpoints of a contiguous interval (this is important for geometry of our construction below).

Applying Proposition 1 we extend (C, f) to a minimal system $(C \times S_1, f \times g)$ where g is an irrational rotation of the circle $S_1 = \{(y, z) \in \mathbb{R}^2 : y^2 + z^2 = 1\}$. Denote by a_1 and b_1 the g -images of the points $(0, 1)$ and $(0, -1)$, respectively. Let J_1 be one of the half-circles determined by a_1, b_1 .

The set C is the union of $C_L = \{x \in C : x \leq c_l\}$ and $C_R = \{x \in C : x \geq c_r\}$. Put $C_R^- = C_R - (c_r - c_l)$. Then $C_L \cup C_R^-$ is a Cantor set with $C_L \cap C_R^- = \{c_l\}$. Further put $S_2 = S_1 + (0, 3)$, $a_2 = a_1 + (0, 3)$, $b_2 = b_1 + (0, 3)$ and $J_2 = J_1 + (0, 3)$. Finally,

denote $M = (C_L \times S_1) \cup (C_R^- \times S_2)$. The dynamical system $(C \times S_1, f \times g)$ induces in a natural way a (minimal) dynamical system (M, F) which is topologically conjugate to $(C \times S_1, f \times g)$ and is obtained from $(C \times S_1, f \times g)$ by just replacing $(C_R \times S_1)$ by its translate $(C_R^- \times S_2)$, ‘without changing dynamics’. In the new system (M, F) the map F preserves ‘vertical’ fibres; the fibre over c_l consists of two circles, each of the other fibres is just a circle. Denote by φ the base map of F . It is clear that (M, F) can be considered as a minimal extension of the dynamical system $(C_L \cup C_R^-, \varphi)$ obtained from (C, f) by identifying points c_l and c_r . Let $\Gamma = S_1 \cup I \cup S_2$ where $I \subseteq \mathbb{R}^2$ is the ‘vertical’ interval with end-points $(0, 1)$ and $(0, 2)$. Put $E = (C_L \cup C_R^-) \times \Gamma$. Then Γ is a connected graph and E is a graph bundle with fibre Γ .

We claim that the map F can be extended to a continuous fibre-preserving map $G : E \rightarrow E$. We are going to define G . Of course, $G|_M = F$. Further, for $x \in C_L \setminus \{c_l\}$ and $(y, z) \in S_2$ put $G(x, y, z) = F(x, y, z - 3)$ and for $x \in C_R^- \setminus \{c_l\}$ and $(y, z) \in S_1$ put $G(x, y, z) = F(x, y, z + 3)$. So, G is already defined on $(C_L \cup C_R^-) \times (S_1 \cup S_2)$. It remains to define G on $(C_L \cup C_R^-) \times (I \setminus \{(0, 1), (0, 2)\})$. So, fix $x \in C_L \cup C_R^-$. Then $G(\{x\} \times (S_1 \cup S_2)) = \{\varphi(x)\} \times S_i$ for some $i \in \{1, 2\}$. Further, $G(x, 0, 1) = \{\varphi(x)\} \times \{a_i\}$ and $G(x, 0, 2) = \{\varphi(x)\} \times \{b_i\}$. For $1 < z < 2$ let $G(x, 0, z)$ be the point of $\{\varphi(x)\} \times J_i$ such that the length of the sub-arc of $\{\varphi(x)\} \times J_i$ with end-points $\{\varphi(x)\} \times \{a_i\}$ and $G(x, 0, z)$ equals $\pi \cdot (z - 1)$.

Then G maps E continuously onto its unique minimal set M . Here M_{c_l} is the union of two circles and M_b for $b \neq c_l$ is a circle. So, M is not a sub-bundle of E .

Case II: M_b consists of two arbitrarily intersecting circles whose union is a graph.

Before giving such a construction we wish to mention that if E were not required to be a graph bundle, it would be sufficient to consider a skew product minimal map on the pinched torus from [9]. In that example, one fibre is “figure eight” (two circles intersecting in one point), all the other fibres are circles (simple closed curves).

The union $P \sqcup Q$ of disjoint sets will sometimes be denoted by $P \sqcup Q$. We will also keep the notations from Case I. Starting with the minimal system $(C \times S_1, f \times g)$ we are going to produce a fibre-preserving selfmap G^* of a direct product graph bundle $E^* \subseteq \mathbb{R}^3$ with the following properties

1. $E^* = (C_L \cup C_R^-) \times \Gamma^*$
2. $\Gamma^* = S_1 \cup S_1^*$ where S_1 is the “geometrical” circle $y^2 + z^2 = 1$ and S_1^* is a “topological” circle (i.e., a simple closed curve) such that
 - $\emptyset \neq S_1 \cap S_1^* \neq S_1$ has finitely many connected components (just because we want E^* to be a graph bundle, i.e. Γ^* has to be a graph),
 - S_1^* is a subset of the closed disc bounded by the circle S_1 and each radius of S_1 contains exactly one point of S_1^* .
3. $M^* = (C_L \times S_1) \cup (C_R^- \times S_1^*)$ is a minimal set for G^* .

Note that each fibre of M^* consists of one circle, except of $M_{c_l}^*$ which consists of two intersecting circles $\{c_l\} \times \{S_1\}$ and $\{c_l\} \times \{S_1^*\}$. Though the only restrictions for the choice of S_1^* are those in (2), let us explicitly mention three simplest cases:

- (12₁) S_1 and S_1^* intersect just in one point (hence $M_{c_l}^*$ is homeomorphic to the “figure eight”), or
- (12₂) S_1 and S_1^* intersect in an arc ($M_{c_l}^*$ is homeomorphic to the “figure Θ ”), or
- (12₃) S_1 and S_1^* intersect in two points.

So, we have the minimal system $(C \times S_1, F_1)$, where $C = C_L \sqcup C_R$ is a Cantor set on the x -axis with $\max C_L = c_l < c_r = \min C_R$ and $F_1 = f \times g$. We are going to construct (E^*, G^*) as above. Fix S_1^* as in (2). Denote by α the projection of S_1 onto S_1^* along the radii of S_1 (hence α is identity on $S_1 \cap S_1^*$) and by σ the map

$$\sigma(x, y, z) := \begin{cases} (x, y, z), & \text{if } (x, y, z) \in C_L \times S_1, \\ (x, \alpha(y, z)), & \text{if } (x, y, z) \in C_R \times S_1. \end{cases}$$

Then $\sigma : C \times S_1 \rightarrow (C_L \times S_1) \sqcup (C_R \times S_1^*)$ is a homeomorphism and so the map

$$F_1^* := \sigma \circ F_1 \circ \sigma^{-1},$$

being topologically conjugate to F_1 , is a continuous *minimal* selfmap of $(C_L \times S_1) \sqcup (C_R \times S_1^*)$. Since $f(c_l) = f(c_r) = \max C$, the definition of F_1^* gives that

$$F_1^*(\{c_l\} \times S_1) = \{\max C\} \times S_1^* = F_1^*(\{c_r\} \times S_1^*), \quad (2.1)$$

$$F_1^*(c_l, y, z) = \sigma(F_1(c_l, y, z)) = \sigma(F_1(c_r, y, z)) = F_1^*(c_r, \alpha(y, z)) \text{ for } (y, z) \in S_1. \quad (2.2)$$

Then (2.1) and (2.2) imply that

$$F_1^*(c_l, y, z) = F_1^*(c_r, y, z) \in \{\max C\} \times S_1^*, \quad \text{for } (y, z) \in S_1 \cap S_1^*. \quad (2.3)$$

Now the idea is to identify the pairs of points $(c_l, y, z), (c_r, y, z)$ where $(y, z) \in S_1 \cap S_1^*$ with the same F_1^* -images and to produce in such a way a new map F_2^* on a new space M^* . Since we wish to keep under control the geometry of our example, we proceed geometrically. In view of (2.3), the mentioned pairs of points become identified if we translate $C_R \times S_1^*$ by the vector $-(c_r - c_l), 0, 0$. Therefore denote $M^* := (C_L \times S_1) \cup (C_R^- \times S_1^*)$ and let $T : (C_L \times S_1) \sqcup (C_R \times S_1^*) \rightarrow M^*$ be defined by

$$T(x, y, z) := \begin{cases} (x, y, z), & \text{if } (x, y, z) \in C_L \times S_1, \\ (x - (c_r - c_l), y, z), & \text{if } (x, y, z) \in C_R \times S_1^*. \end{cases}$$

As already indicated, due to (2.3) there is a unique continuous map $F_2^* : M^* \rightarrow M^*$ such that the following diagram commutes:

$$\begin{array}{ccc} (C_L \times S_1) \sqcup (C_R \times S_1^*) & \xrightarrow{F_1^*} & (C_L \times S_1) \sqcup (C_R \times S_1^*) \\ T \downarrow & & \downarrow T \\ M^* & \xrightarrow{F_2^*} & M^* \end{array}$$

A straightforward analysis of the map F_2^* shows that a point in $\{c_l\} \times S_1 \subseteq M^*$ and a point in $\{c_l\} \times S_1^* \subseteq M^*$ lying on the same radius of the circle $\{c_l\} \times S_1$ have always the same F_2^* -image:

$$F_2^*(c_l, y, z) = F_2^*(c_l, \alpha(y, z)) \quad \text{whenever } (y, z) \in S_1. \quad (2.4)$$

To finish the study of the properties of F_2^* , notice that F_2^* is fibre-preserving and, being a factor of the minimal map F_1^* , is also minimal.

Now define E^* and Γ^* as in 1. and 2. at the beginning of the proof of Case II. To finish our construction, it is sufficient to extend $F_2^* : M^* \rightarrow M^*$ to a continuous fibre-preserving map $G^* : E^* \rightarrow E^*$. Here is one such extension:

$$G^*(x, y, z) := \begin{cases} F_2^*(x, y, z), & \text{if } (x, y, z) \in M^*, \\ F_2^*(x, \alpha^{-1}(y, z)), & \text{if } (x, y, z) \in (C_L \setminus \{c_l\}) \times S_1^*, \\ F_2^*(x, \alpha(y, z)), & \text{if } (x, y, z) \in (C_R^- \setminus \{c_l\}) \times S_1. \end{cases} \quad (2.5)$$

The definition is correct. In fact, the first and the second case are compatible, because if $(x, y, z) \in M^*$ and simultaneously $x \in C_L \setminus \{c_l\}$ and $(y, z) \in S_1^*$, then $(y, z) \in S_1 \cap S_1^*$ and so $(\alpha^{-1})(x, y, z) = (x, y, z)$. Analogously, the first and the third case are compatible. Hence, $G^* : E^* \rightarrow E^*$ is a well defined extension of F_2^* . It is obviously fibre-preserving. To show that it is continuous, it is sufficient to show that the restrictions of G^* to the *closed* sets $C_L \times (S_1 \cup S_1^*)$ and $C_R^- \times (S_1 \cup S_1^*)$ are continuous. Since the arguments for both cases are analogous, we prove only the continuity of G^* on the former set. It is the union of two *closed* sets $C_L \times S_1$ and $C_L \times S_1^*$ and so the continuity of $G^*|_{C_L \times (S_1 \cup S_1^*)}$ follows from the following two facts:

- On the set $C_L \times S_1$, since it is a subset of M^* , the map G^* is continuous because it coincides there, by (2.5), with the continuous map F_2^* .
- On the set $C_L \times S_1^*$ the map G^* is also continuous, because it coincides there with the continuous map $F_2^* \circ (\text{id}_{C_L} \times \alpha^{-1})$ where id_{C_L} is the identity on C_L . To see this, first notice that for $x \in C_L \setminus \{c_l\}$ the coincidence works by (2.5). Further, if $(y^*, z^*) \in S_1^*$ then $(c_l, y^*, z^*) \in M^*$ and so, using (2.5) and (2.4) we get $G^*(c_l, y^*, z^*) = F_2^*(c_l, y^*, z^*) = F_2^*(c_l, \alpha^{-1}(y^*, z^*))$, as required.

The construction is now completed. In the case (12₁) it gives the space E^* made of two “tubes” $(C_L \cup C_R^-) \times S_1$ and $(C_L \cup C_R^-) \times S_1^*$, the second tube lying “inside” the first one and so they touch “internally”. If one wishes that they touch “externally”, i.e. that $M_{c_l}^*$ is a geometric, not only topological “figure eight”, it is sufficient to use an appropriate conjugacy. Similarly, in (12₂) the tubes can “touch externally” along an arc. Also in (12₃) we can get that S_1^* is not anymore a subset of the closed disc bounded by the circle S_1 , but S_1 and S_1^* are two geometric circles having two points in common. \square

3 Dynamical and topological preliminaries

For convenience of the reader, we collect below several dynamical and topological facts which will be used throughout the rest of the paper. The reader should at least pay attention to the concepts of a redundant open set and the homeo-part of a minimal system since they are instrumental in the paper.

3.1 Some basic facts on minimality

In this subsection we always assume that X is a compact metric space and $f : X \rightarrow X$ is a continuous map. The facts here, if not obvious, are mostly results from our

paper [24]. An exception is the equivalence $(1) \Leftrightarrow (3)$ in the below list of equivalent definitions of minimality, which is [8, Lemma 3.10]. For the equivalence $(1) \Leftrightarrow (2)$ involving backward orbits one needs to see the proof of Theorem 2.8 in [24] (cf. [30]).

In the introduction we gave two equivalent definitions of minimality (in terms of invariant subsets and in terms of density of forward orbits). For a compact metric space X and a continuous map $f : X \rightarrow X$ also the following are equivalent:

- (1) (X, f) is minimal,
- (2) $f(X) = X$ and every backward orbit of every point in X is dense (by a *backward orbit* of $x_0 \in X$ we mean any set $\{x_0, x_1, \dots, x_n, \dots\}$ with $f(x_{i+1}) = x_i$ for $i \geq 0$),
- (3) the only closed subsets A of X with $f(A) \supseteq A$ are \emptyset and X ,
- (4) for every non-empty open set $U \subseteq X$, there is $N \in \mathbb{N}$ such that $\bigcup_{n=0}^N f^{-n}(U) = X$.

We will also need some necessary conditions for minimality. If (X, f) is minimal then

- (a) for every non-empty open set $U \subseteq X$, there is $N \in \mathbb{N}$ such that $\bigcup_{n=0}^N f^n(U) = X$,
- (b) f is *feebly open*, i.e. it sends non-empty open sets to sets with non-empty interior,
- (c) f is *almost one-to-one*, which means that the set $\{x \in X : \text{card } f^{-1}(x) = 1\}$ is a G_δ -dense set in X ,
- (d) if $A \subseteq X$ is nowhere dense (dense, of 1st category, of 2nd category, residual) then both $f(A)$ and $f^{-1}(A)$ are nowhere dense (dense, of 1st category, of 2nd category, residual), respectively.

A set $G \subseteq X$ is said to be a *redundant open set for a map* $f : X \rightarrow X$ if G is nonempty, open and $f(G) \subseteq f(X \setminus G)$ (i.e., its removal from the domain of f does not change the image of f).

Lemma 1 ([24]) *Let X be a compact metric space and $f : X \rightarrow X$ continuous. Suppose that there is a redundant open set for f . Then the system (X, f) is not minimal.*

3.2 Homeo-part of a minimal system

Definition 3 Let f be a continuous selfmap of a compact metric space X . Let $H \subseteq X$ be the set of all points $x_0 \in X$ whose full orbit $\{x \in X : \exists i, j \geq 0 \text{ with } f^i(x) = f^j(x_0)\}$ is of the form $\{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$ where $f(x_n) = x_{n+1}$ for every integer n . Then the system $(H, f|_H)$ is said to be the *homeo-part of the system* (X, f) . We also shortly say that H is the homeo-part of f .

One can show that H is always a G_δ set (possibly empty). For minimal maps this is easier to prove and we can say even more.

Lemma 2 *Let X be a compact metric space and $f : X \rightarrow X$ be a minimal map. Then the homeo-part H of f is a dense G_δ set.*

Proof Set $D = \{x \in X : \text{card } f^{-1}(x) > 1\}$. By [24, Theorem 2.8], the homeo-part of a minimal map is residual and D is an F_σ -set of first category. It is straightforward to check that $H = X \setminus \bigcup_{n=-\infty}^{+\infty} f^n(D)$. By (d) from Subsection 3.1 we get that H is G_δ . \square

Lemma 3 *Let X be a compact metric space and $f : X \rightarrow X$ be a continuous minimal map. Let a set $D = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$ be such that $f(x_n) = x_{n+1}$ for every integer n (i.e., D is a union of the forward orbit of x_0 and one of the backward orbits of x_0). Suppose that there is a point in D that has more than one f -preimage in X (or, equivalently, an f -preimage in $X \setminus D$). Then $(f|_D)^{-1}$ is not continuous.*

Proof Suppose, on the contrary, that $g := (f|_D)^{-1} : D \rightarrow D$ is continuous. Without loss of generality we may assume that the mentioned point with two preimages is x_0 . Denote by z a point in $X \setminus D$ with $f(z) = x_0$. Choose two disjoint open neighborhoods U_{-1} and U_z of the points x_{-1} and z , respectively. Denote $V_{-1} := U_{-1} \cap D$. Due to the continuity of g at the point x_0 , we can find a neighborhood U_0 of x_0 such that for $V_0 := U_0 \cap D$ we have $g(V_0) \subseteq V_{-1}$. Now use the continuity of f at the point z to get an open neighborhood $U_z^* \subseteq U_z$ of z with $f(U_z^*) \subseteq U_0$. Since f is minimal, there is $k > 0$ with $x_k \in U_z^*$, whence $x_{k+1} = f(x_k) \in V_0$. Then $g(x_{k+1}) = x_k \in U_z^*$ which contradicts the facts that $g(V_0) \subseteq V_{-1}$ and U_z^* is disjoint with V_{-1} . \square

The next description of properties of the homeo-part of a minimal map follows partially from Theorem 2.8 and its proof in [24]. Note that the notion of a full orbit of a point (for a not necessarily invertible map) was introduced in Definition 3.

Lemma 4 *Let X be a compact metric space and $f : X \rightarrow X$ be a continuous minimal map. Let $H \subseteq X$ be the homeo-part of f . Then:*

- (1) $f(H) = H = f^{-1}(H)$ (equivalently, H is a union of full orbits of the map f),
- (2) every point of the set H has just one f -pre-image (and this pre-image lies in H),
- (3) both $f|_H$ and $(f|_H)^{-1}$ are minimal homeomorphisms $H \rightarrow H$,
- (4) H is a G_δ dense subset of X ,
- (5) H is a maximal subset of X with the properties (1) and (2),
- (6) H is a maximal subset of X with the property (3).

Proof The equivalence in (1) is obvious. The properties (1) and (2) follow from the definition of the homeo-part, see Definition 3. For the property (3) see Theorem 2.8 in [24] and its proof. Lemma 2 gives (4). The property (5) is obvious since if we add something to H , we have to add another full orbit (because we want (1)). This full orbit contains, due to the definition of the homeo-part, a point with two preimages. Then the enlarged set will not satisfy (2). Similarly, Lemma 3 shows that if we add something to H then the enlarged set will not satisfy (3) and so we get (6). \square

Lemma 5 *Let X be a compact metric space and $f : X \rightarrow X$ be minimal. Let H be the homeo-part of f and P be a residual set in X . Then there is a set $R \subseteq X$ such that*

- (1) $R \subseteq P \cap H$ and R is residual in X ,
- (2) $f(R) = R = f^{-1}(R)$,
- (3) both $f|_R$ and $(f|_R)^{-1}$ are minimal homeomorphisms $R \rightarrow R$.

In particular, the inclusion $R \subseteq H$ and (2) give that R is a union of some of the full (i.e. forward and backward) orbits of the homeomorphism $f|_H$.

Proof Put $R = H \cap \bigcap_{n \in \mathbb{Z}} f^n(P)$. The minimal map f preserves residuality in both forward and backward direction. Therefore the set R , being the intersection of countably many residual sets, is residual in X . The rest is obvious. \square

3.3 Locally closed sets and a generalization of Baire category theorem

A subset S of a topological space X is *locally closed* if every $x \in S$ has a neighborhood U such that the intersection $S \cap U$ is closed in the subspace U of X . The following conditions are equivalent, see e.g. [12, p. 112]:

- (1) The set S is locally closed.
- (2) The difference $\overline{S} \setminus S$ is closed (i.e. S is open in \overline{S}).
- (3) S is a difference of two closed sets (intersection of a closed set with an open set).

Lemma 6 *Let X be a topological space and $S \subseteq X$ a locally closed set. If S is not nowhere dense then S has nonempty interior.*

Proof By the assumption (if Int_X denotes the interior in X) we have $V := \text{Int}_X(\overline{S}) \neq \emptyset$. The set V is open in X and hence, being a subset of \overline{S} , obviously also open in \overline{S} . Now the fact that S is dense in \overline{S} gives that $S \cap V \neq \emptyset$. Further, since S is locally closed, S is open in \overline{S} . Therefore there is a set U open in X such that $S = U \cap \overline{S}$. Since $S \subseteq U$ and $S \cap V \neq \emptyset$, we have $U \cap V \neq \emptyset$. This set is open in X and since $U \cap V \subseteq U \cap \overline{S} = S$ we get that $\text{Int}_X(S) \neq \emptyset$. \square

Recall that a *Baire space* is a topological space having the property that whenever a countable union of closed sets has nonempty interior then one of them has nonempty interior (i.e. so called Baire category theorem works). The following lemma gives a generalization of Baire category theorem: it shows that closed sets can be replaced by locally closed ones.

Lemma 7 *Let X be a Baire topological space and $\{S_\lambda : \lambda \in \Lambda\}$ a countable family of subsets of X . Assume that*

- (i) $\bigcup_{\lambda \in \Lambda} S_\lambda$ has nonempty interior in X and
- (ii) for every $\lambda \in \Lambda$, S_λ is locally closed.

Then there is $\lambda_0 \in \Lambda$ such that S_{λ_0} has nonempty interior in X .

Proof By applying Baire category theorem to the closed sets $\overline{S_\lambda}$, $\lambda \in \Lambda$, we get that there is $\lambda_0 \in \Lambda$ such that $\overline{S_{\lambda_0}}$ has nonempty interior. So, S_{λ_0} is not nowhere dense and since it is locally closed, it has nonempty interior in X by Lemma 6. \square

4 Strongly star-like interior points

We introduce the notion of a *strongly* star-like interior point which is more restrictive than that of a star-like interior point of M and, though not appearing in the statement of Theorem A, will play a key role in the proof of it.

First of all recall that, when speaking on a graph bundle, we always assume that it is a (compact) *metric space*, as it was already said in Introduction. To avoid cumbersome formulations, we will often make no distinction between homeomorphic spaces. If (E, B, p, Γ) is a graph bundle and $Q \subseteq E$ and $Z \subseteq \Gamma$, then we say that Q is *canonically homeomorphic* to $U \times Z$, if $p(Q) = U$ and there is a homeomorphism

$h : Q \rightarrow U \times Z$ such that on Q we have $\text{pr}_1 \circ h = p$ (here h is said to be a *canonical homeomorphism*). Notice that, in this terminology, in the above definition of the fibre bundle it is required that $p^{-1}(U)$ be canonically homeomorphic to $U \times \Gamma$.

Recall that if (E, B, p, Γ) is a graph bundle and $M \subseteq E$ and $b \in B$, then the fibre of M over b is $M_b = M \cap p^{-1}(b)$. Further, by Γ_b we will denote the set $p^{-1}(b)$, the fibre over b (now we slightly abuse the already adopted notation M_b , since Γ is not a subset of E). Note that $\Gamma_b = E_b \subseteq E$ is a graph homeomorphic to Γ and if $E = B \times \Gamma$ then $\Gamma_b = \{b\} \times \Gamma$. Also subsets of Γ_b will be sometimes denoted by, say, I_b, T_b , etc. We believe that this will not cause any misunderstanding because always when using notation like X_b it will be clear what kind of a set it is. Recall also that if $M \subseteq E$ and $U \subseteq B$, we denote $M_U = M \cap p^{-1}(U)$.

By an arc we mean a homeomorphic image of a compact real interval. Sometimes we call it a closed arc, since in an obvious way we also use the notions of an open or a half-closed arc. For $N \geq n \geq 2$ let $\Sigma_n \subseteq \Sigma_N$ be two open stars with the same central point. Suppose that Σ_n is the union of some of the half-closed branches of Σ_N (i.e., Σ_n is obtained from Σ_N by removing $N - n \geq 0$ open branches of Σ_N). Then we will say that Σ_n is a *full sub-star* of Σ_N . Here ‘full’ does not mean that $n = N$; it just refers to the fact that Σ_n consists of ‘whole’ branches of Σ_N (rather than of just subsets of them) and so it can be $n < N$. Note also that we consider only the case when $N \geq n \geq 2$ (though, formally, such a definition would make sense for $N \geq n \geq 1$).

Definition 4 Suppose that M is a closed subset of a *product* graph bundle $E = B \times \Gamma$. Then we define $\text{Sint}_s(M)$, the set of strongly star-like interior points of M , as follows. A point $x = (x_1, x_2) \in M$ is said to be a *strongly star-like interior point* of M , if

- x has order $N \geq 2$ in the graph $\Gamma_{x_1} = \{x_1\} \times \Gamma$ (so, $\text{ord}(x_2, \Gamma) = N \geq 2$), and
- there exists an E -open neighborhood $O \times \Sigma_N$ of x such that x_2 is the central point of Σ_N and the corresponding M -open neighborhood $\mathcal{G} = M \cap (O \times \Sigma_N)$ of x has the following structure:

$\mathcal{G}_{x_1} = \{x_1\} \times \Sigma_k$ where $k \geq 2$ and Σ_k is a full sub-star of Σ_N , and for every $z \in p(\mathcal{G}) \subseteq O$ we have $\mathcal{G}_z = \{z\} \times \Sigma_{k(z),z} \subseteq \{z\} \times \Sigma_k$, where $k(z) \in \{2, \dots, k\}$ and $\Sigma_{k(z),z}$ is a full sub-star of Σ_k . (Notice that $\Sigma_{k(x_1),x_1} = \Sigma_k$.) We will say that \mathcal{G} is a *canonical $\text{Sint}_s(M)$ -neighborhood* of x (note that, among others, $\mathcal{G} \subseteq \text{Sint}_s(M)$).

Above, $\text{Sint}_s(M)$ was defined for a closed subset M of $E = B \times \Gamma$. Since each graph bundle is locally trivial and the above definition has a local character, the concept of a strongly star-like interior point has an obvious extension to the case when the graph bundle E is not a direct product space. For a closed set M in an arbitrary graph bundle we set $\text{End}_s(M) = M \setminus \text{Sint}_s(M)$.

Example 5 Let $E = B \times \Gamma$ where $B = [0, 1]$ and $\Gamma = ([-1, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$. Put $A = [0, 1] \times [-1, 1] \times \{0\}$ and

$$\begin{aligned} M^1 &= A \cup \{(x, 0, x) : x \in [0, 1]\}, & M^2 &= M^1 \cup \{(0, 0, z) : z \in [0, 1]\}, \\ M^3 &= A \cup \{(0, 0, z) : z \in [0, 1]\}, & M^4 &= A \cup \{(x, 0, 1-x) : x \in [0, 1]\}. \end{aligned}$$

Then $(0, 0, 0) \notin \text{Sint}_s(M^i)$ for $i = 1, 2$ and $(0, 0, 0) \in \text{Sint}_s(M^i)$ for $i = 3, 4$. \square

In the definition we write $\Sigma_{k(z),z}$ rather than $\Sigma_{k(z)}$ because it may happen that $\Sigma_{k(z_1),z_1}$ and $\Sigma_{k(z_2),z_2}$, considered as subgraphs of Γ , are different even when $k(z_1) = k(z_2)$. The following instructive example illustrates this fact.

Example 6 Let $E = B \times S_4$ where $B = [0, 1]$. Let $(C_n)_{n=1}^\infty$ be a sequence of pairwise disjoint Cantor sets in $(0, 1]$ converging, in the Hausdorff metric, to the singleton $\{0\}$. Denote three of the four closed branches of S_4 by J_1, J_2, J_3 and the central point of S_4 by c . Let M be the set with

$$M_x = \begin{cases} \{x\} \times (J_1 \cup J_2 \cup J_3) & \text{if } x = 0 \\ \{x\} \times (J_1 \cup J_2) & \text{if } x \in C_n \text{ for } n \equiv 1 \pmod{3}, \\ \{x\} \times (J_2 \cup J_3) & \text{if } x \in C_n \text{ for } n \equiv 2 \pmod{3}, \\ \{x\} \times (J_3 \cup J_1) & \text{if } x \in C_n \text{ for } n \equiv 0 \pmod{3}, \\ \emptyset & \text{otherwise,} \end{cases}$$

see Fig.2. Then M is compact and $\{0\} \times \{c\} \in \text{Sint}_s(M)$. In fact all the points of M except of the end-points of the stars M_x , $x \in p(M)$, belong to $\text{Sint}_s(M)$. \square

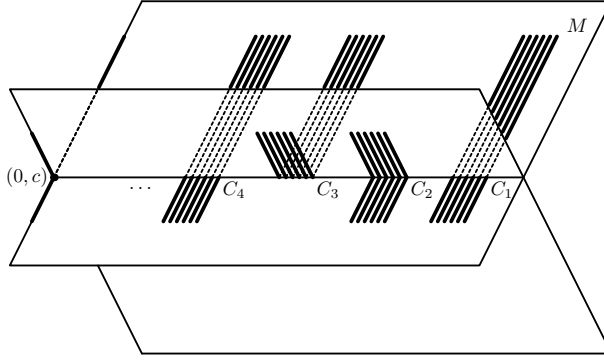


Fig. 2 $(0; c)$ is a strongly star-like interior point of M .

Notice that $\text{Sint}_s(M)$ is open in M (but not necessarily in E) and $\text{End}_s(M)$ is closed in M (hence closed in E). By comparing Definitions 2 and 4 observe that

$$\text{Sint}_s(M) \subseteq \text{Sint}(M) = \bigcup_{b \in B} \text{Sint}(M_b), \quad \text{End}_s(M) \supseteq \text{End}(M) = \bigcup_{b \in B} \text{End}(M_b). \quad (4.1)$$

In general neither of these two inclusions is an equality. For $M \subseteq E$ and $b \in B$ we will further use the notation

$$M_b^{S_s} = M_b \cap \text{Sint}_s(M) = (\text{Sint}_s(M))_b.$$

Example 7 Let $E = B \times \Gamma$ with $B = [-1, 1]$ and $\Gamma = [0, 3]$. Let C be a Cantor set with $\min C = 0$, $\max C = 1$ and let $M = ([-1, 0] \times [0, 1]) \cup (C \times [1, 2]) \cup (\{0\} \times [2, 3])$. Then $\text{Sint}_s(M) = ([-1, 0] \times (0, 1)) \cup (C \times (1, 2)) \cup (\{0\} \times (2, 3))$. So, $M_0^{S_s} = \{0\} \times ((0, 1) \cup (1, 2) \cup (2, 3))$ while $\text{Sint}(M_0) = \{0\} \times (0, 3)$. \square

Lemma 8 *Let (E, B, p, Γ) be a compact graph bundle and $M \subseteq E$ a compact set. Then*

$$\text{End}_s M = \overline{\text{End} M}.$$

Proof Without loss of generality we may assume that $E = B \times \Gamma$. One inclusion is trivial by (4.1). To prove the other one, suppose that there is a point $x \in \text{End}_s(M) \setminus \overline{\text{End} M}$. Then, if the second coordinate of x has order m in Γ , we have $m \geq 2$ (otherwise x would be in $\text{End}(M)$) and some E -open neighborhood $O \times \Sigma_m$ of x is disjoint with $\text{End}(M)$. Hence, if $z \in O$ then the set $(\{z\} \times \Sigma_m) \cap M$ is empty or is of the form $\{z\} \times \Sigma_{k(z), z}$ where $k(z) \in \{2, \dots, m\}$ and $\Sigma_{k(z), z}$ is a full sub-star of Σ_m (otherwise it would necessarily contain a point from $\text{End}(M_z)$). It follows that $x \in \text{Sint}_s(M)$, a contradiction. \square

Lemma 9 *Let (E, B, p, Γ) be a compact graph bundle and $M \subseteq E$ a compact set. If $\text{End}_s(M) = M$ then M is nowhere dense in E .*

Proof If M is somewhere dense in E then, being closed, has nonempty interior in E . It is clear that this interior contains a point which belongs to $\text{Sint}_s(M)$. \square

Lemma 10 *Let (E, B, p, Γ) be a compact graph bundle and $M \subseteq E$ a compact set with $p(M) = B$. If $\text{End}(M) = \emptyset$ then M has nonempty interior in E .*

Proof We may assume that $E = B \times \Gamma$. Let K_1, K_2, \dots, K_k be the list of circles in Γ . For $i = 1, 2, \dots, k$, let $B^{(i)}$ be the set of points $b \in B$ such that M_b contains $\{b\} \times K_i$. The set M is closed and so all the sets $B^{(i)}$ are closed. Since $p(M) = B$ and $\text{End}(M) = \emptyset$, we have $B = \bigcup_{i=1}^k B^{(i)}$ and since the metric space B is compact (hence second category), there is $j \in \{1, 2, \dots, k\}$ such that the (closed) set $B^{(j)}$ has nonempty interior. Since Γ is a graph, it follows that M has nonempty interior in E . \square

Trivial examples show that the converse statements to the previous two lemmas are not true.

Lemma 11 *Let $E = B \times \Gamma$ be a compact graph bundle, $M \subseteq E$ a compact set and $a \in B$. Suppose that $\Delta = \{a\} \times \Delta^\Gamma$ is a compact subset of M_a^{S} . If W is a sufficiently small open neighborhood of a and U is a sufficiently small open neighborhood of Δ^Γ then the E -open neighborhood $W \times U$ of Δ has the following properties:*

- *The corresponding M -open neighborhood $\mathcal{D} = M \cap (W \times U)$ of Δ is a subset of $\text{Sint}_s(M)$.*
- *If we write $\mathcal{D}_z = \{z\} \times \mathcal{D}_z^\Gamma$, then $\mathcal{D}_z^\Gamma \subseteq \mathcal{D}_a^\Gamma$ and $\overline{\mathcal{D}_z^\Gamma} \setminus \mathcal{D}_z^\Gamma \subseteq \overline{\mathcal{D}_a^\Gamma} \setminus \mathcal{D}_a^\Gamma$ whenever $z \in p(\mathcal{D})$.*
- *The set $p(\mathcal{D})$ is closed in W , hence it is a Baire space.*

Proof Since Δ is compact, it can be covered by a finite family of M -open sets $\mathcal{G}^j = M \cap (O_j \times \Sigma_{N(j)})$, $j = 1, \dots, r$, where \mathcal{G}^j are some canonical $\text{Sint}_s(M)$ -neighborhoods of points in Δ . Put $W = \bigcap_{j=1}^r O_j$ and $U = \bigcup_{j=1}^r \Sigma_{N(j)}$. We prove that $\mathcal{D} = M \cap (W \times U)$ satisfies all the requirements. First, it is obvious that \mathcal{D} is an M -open neighborhood of Δ and $\mathcal{D} \subseteq \text{Sint}_s(M)$. Further notice that if we denote, for $j = 1, \dots, r$,

$$M^j = M \cap (W \times \Sigma_{N(j)})$$

then $a \in p(M^j)$, $(M^j)_a = \{a\} \times \Sigma^j$ where Σ^j is a full sub-star of $\Sigma_{N(j)}$ and, for $z \in p(M^j)$, $(M^j)_z = \{z\} \times \Sigma_{j(z),z}$ where $\Sigma_{j(z),z}$ is a full sub-star of $\Sigma^j = \Sigma_{j(a),a}$. Thus

$$\mathcal{D} = M \cap (W \times U) = \bigcup_{j=1}^r M^j = \bigcup_{j=1}^r \bigcup_{z \in p(M^j)} (\{z\} \times \Sigma_{j(z),z}).$$

Fix $z \in p(\mathcal{D}) = \bigcup_{j=1}^r p(M^j)$. Since

$$\mathcal{D}_z^\Gamma = \bigcup_{j=1}^r \Sigma_{j(z),z}, \quad \text{in particular} \quad \mathcal{D}_a^\Gamma = \bigcup_{j=1}^r \Sigma^j,$$

we get $\mathcal{D}_z^\Gamma \subseteq \mathcal{D}_a^\Gamma$. Hence $\overline{\mathcal{D}_z^\Gamma} \subseteq \overline{\mathcal{D}_a^\Gamma}$ and so, to prove that $\overline{\mathcal{D}_z^\Gamma} \setminus \mathcal{D}_z^\Gamma \subseteq \overline{\mathcal{D}_a^\Gamma} \setminus \mathcal{D}_a^\Gamma$, it is sufficient to show that the assumption that some point $q \in \overline{\mathcal{D}_z^\Gamma} \setminus \mathcal{D}_z^\Gamma$ belongs to \mathcal{D}_a^Γ , leads to a contradiction. To this end consider such a point q . Since $q \in \overline{\mathcal{D}_a^\Gamma}$, there is $j \in \{1, \dots, r\}$ such that $q \in \Sigma^j$ and so $q \in U$. On the other hand, $q \in \overline{\mathcal{D}_z^\Gamma}$ and so $(z, q) \in \overline{\mathcal{D}_z} \subseteq \overline{M} = M$. Also, $(z, q) \in W \times U$ because $z \in p(\mathcal{D}) \subseteq W$ and $q \in U$. Thus, $(z, q) \in M \cap (W \times U) = \mathcal{D}$ which implies that $q \in \mathcal{D}_z^\Gamma$, a contradiction.

Now we prove that $p(\mathcal{D})$ is closed in W . It can be seen from the definition that if \mathcal{G} is a canonical $\text{Sint}_s(M)$ -neighborhood of a point $x \in M \subseteq B \times \Gamma$ and for each $z \in p(\mathcal{G})$ we put $\mathcal{G}_z = \{z\} \times \mathcal{G}_z^\Gamma$, then the family $\{\mathcal{G}_z^\Gamma : z \in p(\mathcal{G})\}$ is finite. Since \mathcal{D} was defined using only finitely many such canonical $\text{Sint}_s(M)$ -neighborhoods, we get that also the family $\{\mathcal{D}_z^\Gamma : z \in p(\mathcal{D})\}$ is finite. Therefore, if $p(\mathcal{D}) \ni z_n \rightarrow z \in W$, we may (passing to a subsequence if necessary) assume that all sets $\mathcal{D}_{z_n}^\Gamma$ are the same. But then, since M is closed, obviously \mathcal{D}_z is nonempty and so $z \in p(\mathcal{D})$.

So, the set $p(\mathcal{D})$ is closed (hence is of type G_δ) in the metric space W . Since W is open in B , this implies that $p(\mathcal{D})$ is G_δ in the compact space B . Thus $p(\mathcal{D})$ is a topologically complete (i.e. completely metrizable) space, hence a Baire space (see, e.g., [31, Theorems 12.1 and 9.1]). \square

In the situation from Lemma 11, let $\Delta \subseteq M_a^{Ss}$ be *connected*. Then it is a graph and obviously there exist $m, n \geq 0$ such that every sufficiently small connected open Γ_a -neighborhood V of Δ has the following properties:

- V is connected and (see Lemma 11) $M \cap V \subseteq M_a^{Ss}$,
- $V \setminus \Delta$ consists of pairwise disjoint open arcs $\{a\} \times I_1^\Gamma, \dots, \{a\} \times I_m^\Gamma, \{a\} \times J_1^\Gamma, \dots, \{a\} \times J_n^\Gamma$ where the arcs $\{a\} \times I_i^\Gamma$ are subsets of M_a^{Ss} and the arcs $\{a\} \times J_i^\Gamma$ are disjoint from M_a . Each of these arcs is attached to Δ at an end-point of Δ or at a ramification point of Γ_a (an end-point of Δ can simultaneously be a ramification point of Γ_a).

We extend the notion of a ramification point as follows. If G is a (not necessarily closed and not necessarily connected) subset of a graph Γ and $g \in G$, we say that g is a *ramification point* of G if there is a G -open neighborhood of g which has the form of an open r -star with $r \geq 3$ and with central point g .

By an *open graph* we mean a graph without its end-points if it has any. So, since a graph is a union of finitely many connected graphs, an open graph is a union of finitely

many connected open graphs, whose closures are pairwise disjoint. Notice that, by this definition, a graph having no end-points (in particular, a circle) is also an open graph and that a circle with one point removed is not an open graph. If an open graph G is a subset of a graph Γ then G need not be an open set in Γ . Each ramification point of G is a ramification point of Γ but the converse is not true in general. If Γ is a graph and $G \subseteq \Gamma$ is an open graph, by the end-points of G we mean the end-points of the (closed) graph \overline{G} . It follows from the definition of strongly star-like interior points that the set $M_a^{S_s}$ is open in the topology of M_a (though not necessarily open in the topology of Γ_a). Its connected components are not necessarily open graphs. For instance, $M_a^{S_s}$ can be a circle with one point removed. In any case, $M_a^{S_s}$ is a subset of Γ_a and so the notion of a ramification point can be applied to it.

In the following two technical lemmas we keep the notation from Lemma 11.

Lemma 12 *Let $E = B \times \Gamma$ be a compact graph bundle, $M \subseteq E$ a compact set and $a \in B$. Suppose that $\Delta = \{a\} \times \Delta^\Gamma \subseteq M_a^{S_s}$ is an arc or a circle and does not contain any ramification point of $M_a^{S_s}$. Then for any sufficiently small open neighborhood W of a and any sufficiently small connected open neighborhood U of Δ^Γ as in Lemma 11, it holds that $\mathcal{D} = W^* \times U^*$, i.e. \mathcal{D} has the structure of a direct product. Here $a \in W^* \subseteq W$ is some not necessarily B -open set. If Δ is a circle then $\{a\} \times U^*$ coincides with Δ and if Δ is an arc then $\{a\} \times U^*$ is an open arc containing Δ (and still containing no ramification point of $M_a^{S_s}$).*

Proof Every point from Δ has an M_a -neighborhood in the form of an open arc and so, since $\Delta \subseteq \text{Sint}_s(M)$, Δ can be covered by a finite family of canonical $\text{Sint}_s(M)$ -neighborhoods of points from Δ which have the form (see the proof of Lemma 11)

$$\mathcal{G}^j = M \cap (O_j \times \Sigma_{N(j)}) = V^j \times \Sigma_2^j.$$

Here V^j is a (not necessarily B -open) set containing a and Σ_2^j is an open arc in Γ such that $\{a\} \times \Sigma_2^j \subseteq \text{Sint}_s(M)$ contains no ramification point of $M_a^{S_s}$.

If two open arcs Σ_2^j and Σ_2^i intersect and $z \in \cap O_j$ then $z \in V^j$ if and only if $z \in V^i$. This together with the fact that Δ is connected gives that if $z \in \cap O_j$ then z belongs to all of the sets V^j whenever it belongs to one of them. Now let U be any sufficiently small connected open neighborhood of Δ^Γ so that $(\{a\} \times U) \cap M_a \subseteq \{a\} \times \cup \Sigma_2^j$. Further, let $W \subseteq \cap O_j$ be any open neighborhood of a . Then the claim holds with $U^* = U \cap \cup \Sigma_2^j$ and $W^* = \{z \in W : z \in V^j \text{ for some (hence for all) } j\}$. \square

Example 8 Let us return to Example 6. Denoting by Δ an arc in M_0 containing the ramification point c , we see that without assuming that Δ contains no ramification point of $M_a^{S_s}$, in Lemma 12 one cannot ensure the existence of \mathcal{D} in the form of a direct product. Further, if Δ does not contain the ramification point c and is a sub-arc of, say, J_1 we see that one cannot claim that W^* exists in the class of B -open sets. \square

Example 9 Let us return to Example 7 and put $\Delta = \{0\} \times \{1/2, 3/2, 5/2\}$. Then $\Delta \subseteq M_0^{S_s}$ and it does not contain any ramification point of $M_0^{S_s}$ (even any ramification point of Γ_0). However, Δ is disconnected and there is no M -open neighborhood of Δ of the product form $W^* \times U^*$. \square

Lemma 13 *Let $E = B \times \Gamma$ be a compact graph bundle, $M \subseteq E$ a compact set and $a \in B$. Suppose that $\Delta = \{a\} \times \Delta^\Gamma \subseteq M_a^{S_s}$ is a graph possibly degenerate to a singleton (and possibly containing ramification points of $M_a^{S_s}$, which may or may not be ramification points of Δ). Then for any sufficiently small open neighborhood W of a and any sufficiently small connected open neighborhood U of Δ^Γ as in Lemma 11, the following holds.*

- $(\{a\} \times (U \setminus \Delta^\Gamma)) \cap M \subseteq M_a^{S_s}$ is empty or consists of pairwise disjoint open arcs $\{a\} \times I_1^\Gamma, \dots, \{a\} \times I_m^\Gamma$ ($m \geq 0$ being finite and independent on U , since U is small enough; $m = 0$ means that the described set is empty).
- For each $i = 1, \dots, m$ the open arc $\{a\} \times I_i^\Gamma$ is attached to Δ at a point $p_i = (a, p_i^\Gamma)$ which is an end-point of Δ or a ramification point of $M_a^{S_s}$ (an end-point of Δ can simultaneously be a ramification point of $M_a^{S_s}$ and it can be $p_i = p_j$ even if $i \neq j$), and at each of the end-points of Δ there is at least one such open arc attached to it. Here for every i , the closure of $\{a\} \times I_i^\Gamma$ is an arc and any two of the sets $\Delta, \overline{\{a\} \times I_i^\Gamma}$, $i = 1, \dots, m$ are either disjoint or intersect only at one of the ‘attaching’ points p_i .
- $\mathcal{D}_a = \Delta$ or $\mathcal{D}_a = \Delta \cup \bigcup_{i=1}^m (\{a\} \times I_i^\Gamma)$, depending on whether $m = 0$ or $m \geq 1$. So, \mathcal{D}_a is an open graph.
- The structure of the corresponding M -open neighborhood $\mathcal{D} = M \cap (W \times U) \subseteq \text{Sint}_s(M)$ of Δ is such that for any $z \in p(\mathcal{D})$, \mathcal{D}_z^Γ is a union of finitely many open graphs whose closures are pairwise disjoint, $\mathcal{D}_z^\Gamma \subseteq \mathcal{D}_a^\Gamma$ and $\text{End}(\overline{\mathcal{D}_z^\Gamma}) \subseteq \text{End}(\overline{\mathcal{D}_a^\Gamma})$.
- For any $z \in p(\mathcal{D})$, each of the connected components of \mathcal{D}_z is the union of a (nonempty, closed) possibly degenerate subgraph of $\{z\} \times \Delta^\Gamma$ and some (possibly zero) of the open arcs $\{z\} \times I_i^\Gamma$ with the ‘attaching’ points (z, p_i^Γ) belonging to \mathcal{D}_z . If this subgraph is nondegenerate and does have one or more end-points, then at each of these end-points there is at least one of these open arcs attached to it. If the subgraph is a singleton (which may happen even if Δ is nondegenerate) then at least two of these open arcs are attached to it.

In particular, if Δ is a tree, possibly degenerate to a singleton, then:

- For each $z \in p(\mathcal{D})$, the set \mathcal{D}_z contains (a nonempty closed subgraph of $\{z\} \times \Delta^\Gamma$, possibly disconnected, possibly degenerate to a finite set, and) at least two of the open arcs $\{z\} \times I_i^\Gamma$, with the ‘attaching’ points (z, p_i^Γ) belonging to \mathcal{D}_z .

Of course, if Δ is a singleton, then the last statement of the lemma does not say anything more than the definition of a strongly star-like interior point of M .

Proof The arguments are completely analogous to those used in the proof of Lemma 12. In fact, the first three parts are just consequences of our definitions of $M_a^{S_s}$, ramification points, endpoints and open graphs. The rest follows from Lemmas 11, 12 and the remarks above Lemma 12. (Note that a key role is played by the fact that $\mathcal{D} \subseteq \text{Sint}_s(M)$). For instance, if the intersection of \mathcal{D}_z with $\{z\} \times \Delta^\Gamma$ is a singleton, then at least two open arcs have to be attached to this singleton, otherwise \mathcal{D}_z could not be a subset of $\text{Sint}_s(M)$. \square

5 Proof of Theorem A

We will use the notation $F_z = F|_{I_z}$. So, F_z is a map from I_z into $F_{f(z)}$.

We start with the following result partially describing F on its minimal sets in case (A2) of our Theorem A. Its use simplifies arguments in the proof of Theorem A.

Proposition 2 *Let the assumptions of Theorem A be satisfied. Let I_a be a closed arc and T_b be a tree such that $I_a \subseteq M_a^{S_s}$, $T_b \subseteq M_b$ and $F(I_a) \subseteq T_b$. If the interior of I_a does not contain any ramification point of $M_a^{S_s}$ then $F|_{I_a}$ is monotone (hence $F(I_a)$ is an arc or a point).*

The statement in the parentheses is obvious since a monotone image of an arc cannot be a nondegenerate tree. Both cases (i.e., $F(I_a)$ is an arc or a point) occur in the example of a noninvertible fibre-preserving minimal map on the torus in [24] (the base is a ‘horizontal’ circle, the fibres are ‘vertical’ circles). Since in this example there is a vertical arc mapped by F into a point while the vertical circle containing this arc is mapped onto a circle, the example also shows that the proposition would not be true if T_b were allowed to contain a circle.

Proof It is sufficient to prove a weaker version of the proposition which is obtained by adding the assumption that neither the end-points of I_a are ramification points of $M_a^{S_s}$. For if one or both end points of I_a are ramification points of $M_a^{S_s}$ then, by applying such a weaker proposition to all sub-arcs J_a of I_a which do not contain end-points of I_a , we get the monotonicity of F on the whole interior of I_a . Since the F -image of this interior is a point or a (not necessarily closed) arc and T_b does not contain a circle, F is obviously monotone on I_a .

So, let I_a contain no ramification point of $M_a^{S_s}$ and suppose, on the contrary, that $F|_{I_a}$ is not monotone. Then there exists $q \in T_b$ such that $(F|_{I_a})^{-1}(q) \subseteq I_a$ is not connected. Take two points u, v in two different connected components of $(F|_{I_a})^{-1}(q)$ and consider the (unique) arc $J_a \subseteq I_a$ with the end-points u, v . From the choice of u, v it follows that there is a point $w \in J_a$ with $F(w) \neq q$. This point w partitions J_a into two nondegenerate closed sub-arcs J_a^1 and J_a^2 . The set $F(J_a) = F_a(J_a) \subseteq T_b$ is a nontrivial continuum (hence a tree) and each of the sets $F(J_a^1)$ and $F(J_a^2)$ contains the (unique) arc in T_b having the end-points $F(w)$ and q . It follows that the arc J_a contains two disjoint closed nondegenerate sub-arcs T_a^1, T_a^2 such that $F(T_a^1)$ and $F(T_a^2)$ are closed arcs with $F(T_a^1) \subseteq \text{Int} F(T_a^2)$ (where by $\text{Int} F(T_a^2)$ we mean the arc $F(T_a^2)$ without its end-points).

Now, since we will work only with some neighborhood of a , we may assume that E has the structure of a product space, i.e. $E = B \times \Gamma$. So I_a has the form $\{a\} \times I$ and similarly $T_a^1 = \{a\} \times T^1$ and $T_a^2 = \{a\} \times T^2$. By Lemma 12, there is an M -open neighborhood \mathcal{D} of I_a which has the product form $\mathcal{D} = W^* \times U^*$ for some (not necessarily B -open) set $W^* \ni a$ and some open arc U^* containing I .

Since $F_a(\{a\} \times T^1) \subseteq \text{Int} F_a(\{a\} \times T^2)$ and since (by replacing T^1 by a smaller arc if necessary) we may assume that the arc $F_a(\{a\} \times T^1)$ does not contain any ramification point of T_b , we have $F_x(\{x\} \times T^1) \subseteq \text{Int} F_x(\{x\} \times T^2)$ also for all x sufficiently close to a . By replacing W^* by its intersection with a small open neighborhood of a

if necessary, we may assume that this is the case for all $x \in W^*$. Then

$$F|_M(W^* \times \text{Int } T^1) \subseteq F|_M(W^* \times T^2) \subseteq F|_M(M \setminus (W^* \times \text{Int } T^1)).$$

Hence the nonempty M -open set $W^* \times \text{Int } T^1$ is redundant for $F|_M$ which contradicts the minimality of $F|_M$.

When $M \subseteq E$ and $\beta \in \text{End}(M)$, i.e. $\beta \in \text{End}(M_b)$ where $b = p(\beta)$, then still it can happen that there is an open arc $J \subseteq M_b$ such that $\beta \in J$ (e.g., let Γ_b be a 3-star S_3 with central point β , M_b be the union of a 2-star S_2 with the same central point β and a sequence of points lying in $S_3 \setminus S_2$ and converging to β). However, the following lemma holds.

Lemma 14 *Let the assumptions of Theorem A be satisfied. Suppose that there exists a point in $\text{End}(M) \setminus F(\text{End}_s(M))$. Then in the same fibre there exists also a point $\beta \in \text{End}(M) \setminus F(\text{End}_s(M))$ such that no open arc containing β exists in M_b , $b = p(\beta)$.*

Proof Choose any $\beta' \in \text{End}(M) \setminus F(\text{End}_s(M))$ and denote $p(\beta') = b$. Suppose that β' is contained in an open arc $J \subseteq M_b$. Then, since $\beta' \notin \text{Sint}(M_b)$, the point β' is necessarily a ramification point of Γ_b and in one of the small open branches emanating from β' there are both a sequence of points in M_b converging to β' and a sequence of points in $\Gamma_b \setminus M_b$ converging to β' . Then this branch obviously contains also a sequence of points $\beta_n \rightarrow \beta'$ such that, for every n , $\beta_n \in \text{End}(M_b)$ and no open arc in M_b contains β_n . Now it is sufficient to put $\beta = \beta_n$ for a sufficiently large n , because $F(\text{End}_s(M))$ is a closed set which does not contain β' .

We are finally ready to prove our Theorem A.

Theorem A. *Let M be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle (E, B, p, Γ) . Then there are two mutually exclusive possibilities:*

- (A1) $\overline{\text{End}(M)} = M$ (and this holds if and only if M is nowhere dense in E);
- (A2) $\text{End}(M) = \emptyset$ (and this holds if and only if M has nonempty interior in E).

In particular, the fibre-preserving maps in tree bundles have only nowhere dense minimal sets.

Proof Also the last claim is obvious, since if Γ is a tree then $\text{End}(M) \neq \emptyset$ and we are therefore in the case (A1). Thus, taking into account Lemmas 8, 9 and 10, it remains to prove the dichotomy: either $\overline{\text{End}(M)} = M$ or $\text{End}(M) = \emptyset$. To this end suppose that $\text{End}(M) \neq \emptyset$. To prove that then $\overline{\text{End}(M)} = M$, it suffices to show that every point in $\text{End}(M)$ has an F -pre-image in $\text{End}_s(M)$. Indeed, suppose for a moment that $F(\text{End}_s(M)) \supseteq \text{End}(M)$. Then $F(\text{End}_s(M)) \supseteq \overline{\text{End}(M)} = \text{End}_s(M)$ by Lemma 8. It follows that the nonempty closed set $\text{End}_s(M)$ is not a proper subset of M , otherwise $(M, F|_M)$ is not minimal, see the equivalence (1) \Leftrightarrow (3) in Subsection 3.1. So, $\text{End}_s(M) = M$ whence by Lemma 8 we get $\overline{\text{End}(M)} = M$.

Thus, to finish the proof, we suppose that there is a point $\beta \in \text{End}(M) \setminus F(\text{End}_s(M))$ and we want to get a contradiction. If we denote $p(\beta) = b$, by Lemma 14 we can assume that

$$\text{there is no open arc in } M_b \text{ containing } \beta. \quad (5.1)$$

Since $F(M) = M$ and $\beta \notin F(\text{End}_s(M))$, there is a point $\alpha \in \text{Sint}_s(M)$ with $F(\alpha) = \beta$. Denote $p(\alpha) = a$. From now on we will work only with neighborhoods of Γ_a and Γ_b and so, due to the local triviality of the graph bundle, we may assume that $E = B \times \Gamma$. Let $\text{ord}(\beta, \Gamma_b) = r \geq 1$, i.e. $\beta = (b, \beta^\Gamma)$ where β^Γ is the central point of an open r -star in Γ . Since the set $F(\text{End}_s(M))$ is closed in E and does not contain β , for some B -open neighborhood O of b and some open r -star Σ_r with the central point β^Γ the open E -neighborhood $\mathcal{O}^* = O \times \Sigma_r$ and hence also the M -open neighborhood $\mathcal{O} = \mathcal{O}^* \cap M$ of β are disjoint from $F(\text{End}_s(M))$. In view of (5.1),

the connected component of $M_b \cap \mathcal{O}$ containing β is either the singleton β
or a (half-closed or closed) arc whose one end-point is β . (5.2)

Recall that $F_z = F|_{\Gamma_z}$. Consider the map $F_a : \Gamma_a \rightarrow \Gamma_b$ and choose that connected component Δ of the set $F_a^{-1}(\beta) \cap M$ which contains the point α . Since $\beta \notin F(\text{End}_s(M))$, we have $\Delta \subseteq \text{Sint}_s(M)$. The set Δ is closed, so it is the singleton α or a (nondegenerate closed) connected subgraph of Γ_a containing α . Let Δ^Γ be the counterpart of Δ in Γ , i.e., $\Delta = \{a\} \times \Delta^\Gamma$.

Let W be a B -open neighborhood of a and U be a connected Γ -open neighborhood of Δ^Γ , both as small as Lemma 13 requires. In what follows, $\mathcal{D} = M \cap (W \times U) \subseteq \text{Sint}_s(M)$, I_i^Γ and $p_i = (a, p_i^\Gamma)$ will have the meaning from this lemma. We will also consider the half-closed arcs $A_i^\Gamma = \{p_i^\Gamma\} \cup I_i^\Gamma$, $i = 1, \dots, m$. Since $F(\Delta)$ is just the singleton β , we may also assume that W and U are small enough to give

$$F(\mathcal{D}) \subseteq \mathcal{O}, \text{ hence none of the sets } F(\mathcal{D}_z), z \in W, \text{ contains a circle.} \quad (5.3)$$

Claim. There is $d \in p(\mathcal{D})$ such that \mathcal{D}_d^Γ contains no circle (and each component of \mathcal{D}_d^Γ is nondegenerate since $\mathcal{D} \subseteq \text{Sint}_s(M)$ and \mathcal{D} is M -open). Moreover, $m \geq 2$ and \mathcal{D}_d contains at least two different half-closed arcs from the list $\{d\} \times A_i^\Gamma$, $i = 1, \dots, m$.

Prof of Claim. Let $C_1^\Gamma, \dots, C_q^\Gamma$, $q \geq 0$, be the list of all (not necessarily pairwise disjoint) circles in Δ^Γ . If $z \in p(\mathcal{D})$ then, by Lemma 13, $\mathcal{D}_z^\Gamma \subseteq \mathcal{D}_a^\Gamma = \Delta^\Gamma \cup \bigcup_{i=1}^m I_i^\Gamma$ and \mathcal{D}_a^Γ contains only those circles which are contained in Δ^Γ . So, if \mathcal{D}_z^Γ contains a circle, it is necessarily a circle from the list $C_1^\Gamma, \dots, C_q^\Gamma$. Denote

$$K_i = \{z \in p(\mathcal{D}) : \mathcal{D}_z^\Gamma \supseteq C_i^\Gamma\}, \quad i = 1, \dots, q.$$

To prove the claim suppose, on the contrary, that for every $z \in p(\mathcal{D})$, \mathcal{D}_z^Γ contains a circle. Then $q \geq 1$ and

$$p(\mathcal{D}) = \bigcup_{i=1}^q K_i.$$

Each of the sets K_i , $i = 1, \dots, q$, is obviously closed in the set $p(\mathcal{D})$ which is, by Lemma 11, a Baire space. Hence there is $s \in \{1, \dots, q\}$ with

$$\text{Int}_{p(\mathcal{D})} K_s \neq \emptyset. \quad (5.4)$$

Now fix an arbitrary $j \in \{1, \dots, q\}$ and an open arc L_j^Γ in C_j^Γ such that the closure of L_j^Γ contains only points of order 2 in Γ (in particular, L_j^Γ has positive distance

from the set $\{p_i^\Gamma : i = 1, \dots, m\}$. Observe that then for every $z \in K_j$ the map F_z is, by Proposition 2 (see also (5.3)), monotone on $\{z\} \times L_j^\Gamma$ and so $F_z(\{z\} \times L_j^\Gamma)$ is an open, closed or half-closed arc, possibly degenerate to a point. Since $F_z(\mathcal{D}_z)$ is by (5.3) a tree (which is a uniquely arcwise connected space), we have that $F_z(\{z\} \times (C_j^\Gamma \setminus L_j^\Gamma)) \supseteq F_z(\{z\} \times L_j^\Gamma)$. Hence

$$F(S \times L_j^\Gamma) \subseteq F(M \setminus (S \times L_j^\Gamma)) \quad \text{for any set } S \subseteq K_j, j \in \{1, \dots, q\}. \quad (5.5)$$

Note also that here $S \times L_j^\Gamma \subseteq M$.

Then by (5.5), for $j = s$ and $S = \text{Int}_{p(\mathcal{D})} K_s$ we obtain $F(\text{Int}_{p(\mathcal{D})} K_s \times L_s^\Gamma) \subseteq F(M \setminus (\text{Int}_{p(\mathcal{D})} K_s \times L_s^\Gamma))$. Therefore, since the set $\emptyset \neq \text{Int}_{p(\mathcal{D})} K_s \times L_s^\Gamma \subseteq M$ is obviously open in the topology of M , the set $\text{Int}_{p(\mathcal{D})} K_s \times L_s^\Gamma$ is a redundant open set for $F|_M$, which contradicts the minimality of $F|_M$. We have thus proved that there exists $d \in p(\mathcal{D})$ such that \mathcal{D}_d^Γ contains no circle.

Applying now the last assertion of Lemma 13, we find that \mathcal{D}_d contains at least two different half-closed arcs from the list $\{d\} \times A_i^\Gamma, i = 1, \dots, m$. Thus $m \geq 2$ which finishes the proof of the claim. $\checkmark\checkmark\checkmark$

Next, we will replace W by a smaller open neighborhood of a and U by a smaller connected open neighborhood of Δ^Γ so that \mathcal{D} have an additional nice property. We are going to show how to do that. Note also that the Claim will still work.

Recall that, by the Claim, $m \geq 2$. The attaching points $p_i = (a, p_i^\Gamma), i = 1, 2, \dots, m$ belong to Δ and so are mapped to the point β . On the other hand, Δ is disjoint with the open arcs $\{a\} \times I_i^\Gamma$. Therefore each of the sets $F(\{a\} \times A_i^\Gamma)$ is a *nondegenerate* connected set in M_b containing β . Taking into account (5.2), we see that each of these sets is in fact a closed or half-closed arc containing β as one of its end-points (so that the connected component of $M_b \cap \mathcal{O}$ containing β is not a singleton, see (5.2)), and $F(\{a\} \times A_i^\Gamma) \subseteq F(\{a\} \times A_j^\Gamma)$ or $F(\{a\} \times A_j^\Gamma) \subseteq F(\{a\} \times A_i^\Gamma)$ whenever $i, j \in \{1, \dots, m\}$. By replacing the half-closed arcs A_i^Γ by shorter ones (i.e., by replacing U by a smaller connected open neighborhood of Δ^Γ) if necessary, we may assume that each of the half-closed arcs $\{a\} \times A_i^\Gamma$ is monotonically (see (5.3) and Proposition 2) mapped by F onto the *same* half closed arc H with the end-point $\beta \in F(\{a\} \times A_i^\Gamma)$ and another end-point $\beta^* \notin F(\{a\} \times A_i^\Gamma)$.

Now fix $k \in \{1, \dots, m\}$ and choose a small open arc $J_k = \{a\} \times J_k^\Gamma$ such that the closure of J_k lies in the interior of $\{a\} \times A_k^\Gamma$ and the closure of $F(J_k)$ lies in the interior of H . Then the closure of $F(\{a\} \times J_k^\Gamma)$ lies in the interior of $F(\{a\} \times A_i^\Gamma)$ for every $i = 1, 2, \dots, m$. By continuity, and replacing W by a smaller neighborhood of a if necessary, we may assume that

$$F(\{z\} \times J_k^\Gamma) \subseteq F(\{z\} \times A_i^\Gamma) \text{ for every } z \in W \text{ and } i = 1, 2, \dots, m. \quad (5.6)$$

Note that this holds (i.e., such a J_k^Γ exists) for any $k \in \{1, \dots, m\}$.

Now we can finish the proof. By the Claim, there exists $d \in p(\mathcal{D})$ such that \mathcal{D}_d does not contain any circle and contains at least two different half-closed arcs, say $\{d\} \times A_1^\Gamma$ and $\{d\} \times A_2^\Gamma$. Both these properties are shared by all the points $z \in p(\mathcal{D})$ sufficiently close to the point d . Indeed, M is closed and Γ contains only finitely many circles and so, if $z \in p(\mathcal{D})$ is close to d , neither the set \mathcal{D}_z can contain a circle. But

then, using the same argument as for the point d (see the very end of the proof of the Claim), the set \mathcal{D}_z also contains at least *two* of the half-closed arcs $\{z\} \times A_i^\Gamma$. It follows that for any $z \in p(\mathcal{D})$ close to d there is at least one $i \neq 1$ such that $\{z\} \times A_i^\Gamma \subseteq M$ and so, regardless of whether $\{z\} \times J_1^\Gamma \subseteq \{z\} \times A_1^\Gamma$ is a subset of M or is disjoint from M , the condition (5.6) applied to $k = 1$ gives $F(M_z \setminus (\{z\} \times J_1^\Gamma)) \supseteq F(M_z \cap (\{z\} \times J_1^\Gamma))$. Hence, for sufficiently small neighborhood $W_1 \subseteq W$ of d we have $F(M \setminus (W_1 \times J_1^\Gamma)) \supseteq F(M \cap (W_1 \times J_1^\Gamma))$ and so the nonempty M -open set $M \cap (W_1 \times J_1^\Gamma)$ is redundant for $F|_M$, a contradiction with minimality of $F|_M$. \square

6 Proof of Theorem C

If $M \subseteq E$ is a closed set with $\text{End}(M) = \emptyset$, we have $\text{End}(M_b) = \emptyset$ for every $b \in B$ and so every set M_b is a (possibly disconnected) graph without end-points (this in particular means that for every $b \in B$ the set M_b contains at least one circle). We will be interested in whether such a graph M_b has a ramification point or not. Of course, M_b does not have any ramification point if and only if it is a union of disjoint circles. Denote

$$\begin{aligned} \mathcal{R}_B(M) &:= \{b \in B : M_b \text{ has a ramification point}\}, \\ \mathcal{R}_E(M) &:= \{\gamma \in E : \gamma \text{ is a ramification point of } M_{p(\gamma)}\}. \end{aligned}$$

Lemma 15 *Let $E = B \times \Gamma$ be a compact graph bundle and $M \subseteq E$ a closed set with $\text{End}(M) = \emptyset$.*

- (a) *If U is an open ball in B with $U \subseteq \mathcal{R}_B(M)$ then there are an open ball $V \subseteq U$ and a ramification point q of Γ such that $V \times \{q\} \subseteq \mathcal{R}_E(M)$.*
- (b) *Let q be a ramification point of Γ of order N and V be an open ball in B with $V \times \{q\} \subseteq \mathcal{R}_E(M)$. Let an open star $\Sigma_N \subseteq \Gamma$ with central point q be a Γ -open neighborhood of q (i.e., Σ_N contains no ramification point of Γ different from q). Then there are a full sub-star Σ_k of Σ_N with $k \geq 3$ and an open ball $W \subseteq V$ in B such that $(W \times \Sigma_N) \cap M = W \times \Sigma_k$ (hence $W \times \Sigma_k$ is an M -open set).*

Proof (a) For each $u \in U$ there is q_u in Γ such that $(u, q_u) \in M$ is a ramification point of M_u . Since there are only finitely many ramification points in Γ and the set M is closed, we get that the same q works for all u in a subset of U with nonempty interior.

(b) For all $v \in V$, (v, q) is a ramification point of M_v . The neighborhood Σ_N of q is a disjoint union of the point q and N open arcs emanating from q . If $v \in V$ and I is one of these open arcs then $\{v\} \times I$ is either a subset of M_v or disjoint with M_v (because the graph M_v , possibly disconnected, has no end-points). Let I_i , $i = 1, \dots, r$ be the list of those of the N open arcs for which $\{v\} \times I_i \subseteq M_v$ for at least one $v \in V$. We say that $v \in V$ has *signature* $\lambda = \{i_1, \dots, i_s\}$ if M_v contains from this list just the open arcs $\{v\} \times I_{i_1}, \dots, \{v\} \times I_{i_s}$. So, the signature λ is a subset (with cardinality at least three) of $\{1, \dots, r\}$. Let Λ be the family of signatures of all points $v \in V$. Then Λ is finite and if S_λ is the set of all points $v \in V$ with signature λ , then $V = \bigcup_{\lambda \in \Lambda} S_\lambda$. Until the end of the proof we will work in (the topology of) the Baire space V . Denote by \overline{S}_λ the closure (in V) of S_λ . We claim that $\overline{S}_\lambda \setminus S_\lambda$ is closed in \overline{S}_λ , i.e., S_λ is locally closed (in V). The reason is as follows. If $x \in \overline{S}_\lambda$ has signature μ then, since M is closed,

$\mu \supseteq \lambda$. If $x \in \overline{S_\lambda} \setminus S_\lambda$ then $\mu \supsetneq \lambda$. This property of having the signature strictly larger than λ is obviously inherited by the limit of a sequence of points from $\overline{S_\lambda} \setminus S_\lambda$. It follows that $\overline{S_\lambda} \setminus S_\lambda$ is closed. So, applying Lemma 7 to the Baire space V we get that there is an open ball W in V (hence W is an open ball in B) such that all points $w \in W$ have the same signature $\{i_1, \dots, i_k\}$ (of cardinality $k \geq 3$). It follows the existence of a full sub-star Σ_k of Σ_N with the required properties. \square

Lemma 16 *Let M be a minimal set (with full projection onto the base) of a fibre-preserving map F in a direct product graph bundle $E = B \times \Gamma$. Assume that $\text{End}(M) = \emptyset$. Suppose that an open ball V in B and a ramification point $q \in \Gamma$ are such that $V \times \{q\} \subseteq \mathcal{R}_E(M)$. Then there are an open ball $V^* \subseteq V$ and a ramification point \tilde{q} of Γ such that $F(V^* \times \{q\}) = f(V^*) \times \{\tilde{q}\} \subseteq \mathcal{R}_E(M)$. The same is true for closed balls instead of open ones.*

Proof Choose Σ_N , Σ_k and W by Lemma 15(b). It is obviously sufficient to show that $F(W \times \{q\}) \subseteq \mathcal{R}_E(M)$. Indeed, then (since Γ has only finitely many ramification points and F is continuous) for any sufficiently small open ball V^* in B such that $V^* \subseteq W$, the second projection of the set $F(V^* \times \{q\}) \subseteq \mathcal{R}_E(M)$ will be just a singleton \tilde{q} (a ramification point of Γ).

So, fix any $a \in W$ (from now on we will write W_a instead of W , to indicate that it contains a) and put $\alpha = (a, q)$, $\beta = F(\alpha) = (b, p)$ (of course, $b = f(a)$ and $\beta \in M = \text{Sint}_s(M)$). We are going to prove that $\beta \in \mathcal{R}_E(M)$.

Suppose, on the contrary, that $\beta \notin \mathcal{R}_E(M)$. Then $\beta \in \text{Sint}_s(M) \setminus \mathcal{R}_E(M)$ and so one can apply Lemma 12 to a small arc in M_b containing β , to obtain that there is an M -open neighborhood of β in the form $W_b \times \Sigma_2$ where W_b contains b but it need not be a B -neighborhood of b and p is the central point of Σ_2 . Recall that W_a is a B -open neighborhood of a and $W_a \times \Sigma_k$ is an M -open neighborhood of $\alpha \in \text{Sint}_s(M)$. Since F is continuous, we may assume that W_a and Σ_k are small enough so that $F(W_a \times \Sigma_k) \subseteq W_b \times \Sigma_2$. We are going to show that there exists a redundant open set for $F|_M$, which will contradict the minimality of $F|_M$. To this end consider two cases.

First assume that there exists $x \in W_a$ such that at least three different (half-closed) branches of $\{x\} \times \Sigma_k$ are mapped by F onto nondegenerate sets, i.e., onto (not necessarily closed) arcs containing the point $F(x, q)$. Then there is a point in $\{f(x)\} \times \Sigma_2$ different from $F(x, q)$ which is F -covered twice, by points P, Q belonging to different branches of $\{x\} \times \Sigma_k$. Hence, some open arc $\{x\} \times J$ in the branch containing P is such that the closure of its F -image lies in the interior (in topology of $M_{f(x)}$) of the F -image of the branch containing Q . Since such a property carries over to all fibres close to the fibre over x , the existence of a redundant open set for $F|_M$ easily follows.

So, for every $x \in W_a$ there are at most two of k branches of $\{x\} \times \Sigma_k$ which are mapped by F to nondegenerate sets. If we denote by J_1, \dots, J_k the branches of Σ_k and by W^i the set of all $x \in W_a$ with $F(\{x\} \times J_i) = F(x, q)$, then W^i is closed in W_a and, since $k \geq 3$, we have $W_a = \bigcup_{i=1}^k W^i$. Since $W_a \times \Sigma_k = \bigcup_{i=1}^k (W^i \times \Sigma_k)$ and the sets $W^i \times \Sigma_k$ are closed in $W_a \times \Sigma_k$, there is i_0 such that $W^{i_0} \times \Sigma_k$ has nonempty interior in $W_a \times \Sigma_k$. It follows that W^{i_0} has nonempty interior in W_a . Thus there is a set $\emptyset \neq \Omega \subseteq W^{i_0}$ open in W_a . So, if A is an open arc lying in J_{i_0} , the set $\Omega \times A$ is open in $W_a \times \Sigma_k$, hence open in M . Since it is redundant for $F|_M$, the proof is finished. \square

Theorem C (full version). *Let M be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle (E, B, p, Γ) . Assume that M has nonempty interior. Then the following holds.*

- (C1) $M = \text{Sint}_s(M)$.
- (C2) *If B is infinite then M exhibits the following kind of ‘perfectness’:*
 - *If U is a trivializing neighborhood, $h : p^{-1}(U) \rightarrow U \times \Gamma$ a canonical homeomorphism and $\tilde{M}_U = h(M_U)$, then for every $(z, p) \in \tilde{M}_U$ there is a sequence of points $U \ni z_n \rightarrow z$, $z_n \neq z$, such that $(z_n, p) \in M_U$ for all n .*
- (C3) $\mathcal{R}_B(M)$ is a closed nowhere dense subset of B .
- (C4) *All the sets M_b , $b \in B$, are unions of circles. In fact there exist an open dense set $\mathcal{O} \subseteq B$ and a positive integer m such that*
 - *for each $z \in \mathcal{O}$, M_z is a disjoint union of m circles, and*
 - *for each $z \in B \setminus \mathcal{O}$, M_z is a union of circles which properly contains a disjoint union of m circles.*

In particular, if M_z is a circle for some $z \in B$, then M_z is a circle for all z in the open dense subset \mathcal{O} of B .
- (C5) *For each $z \in \mathcal{O}$ there exists a trivializing neighborhood $z \in U \subseteq \mathcal{O}$ such that if $h : p^{-1}(U) \rightarrow U \times \Gamma$ is a canonical homeomorphism then $\tilde{M}_U = h(M_U)$ has the structure of a direct product. It means that $\tilde{M}_U = U \times \bigcup_{i=1}^m C_i$ where C_1, \dots, C_m are pairwise disjoint circles in Γ . Consequently,*
 - *if $\mathcal{O} = B$, then M is a sub-bundle of E whose fibre is a disjoint union of m circles, and*
 - *if $\mathcal{O} = B$, $E = B \times \Gamma$ and B is connected, then M is a direct product of B and a disjoint union of m circles.*
- (C6) *The set $M_{\mathcal{O}}$ is dense in M .*
- (C7) *Call a circle $\mathcal{K} \subseteq M_b$, $b \in B$, a generating circle if there are circles $\mathcal{K}_n \subseteq M_{b_n}$, $b_n \in \mathcal{O}$, $n = 1, 2, \dots$, such that $\mathcal{K}_n \rightarrow \mathcal{K}$ with respect to the Hausdorff metric in E . Then the set M is the union of all generating circles. If $b \in \mathcal{O}$ then M_b is a disjoint union of m circles and each of them is in fact generating. If $b \in B \setminus \mathcal{O}$, the set M_b may contain a circle that is not generating but it always contains at least $m+1$ generating circles, at least m of them being pairwise disjoint.*
- (C8) *If $z \in \mathcal{O}$ then the set M_z , which is a disjoint union of m circles, is mapped by F onto a disjoint union of m circles in $M_{f(z)}$.*
- (C9) *If $z \in B \setminus \mathcal{O}$ then a generating circle in M_z is mapped by F onto a generating circle in $M_{f(z)}$. A non-generating circle in M_z need not be mapped onto a circle.*
- (C10) *If f is monotone then $\mathcal{O} = B$ (hence, M is a sub-bundle of E).*
- (C11) *If $E = B \times \Gamma$ and B is locally connected then $\mathcal{O} = B$ (hence, M is a sub-bundle of E and if B is also connected, then M is a direct product).*

Concerning (C8), let us remark that if $z \in \mathcal{O}$ and S is a circle in M_z then the map $F|_S : S \rightarrow M_{f(z)}$ need not be injective even if f is a homeomorphism (see the non-invertible skew-product torus map in [24]) and the map $F|_{M_z} : M_z \rightarrow M_{f(z)}$ need not be surjective (see Theorem D).

In (C9), two different/disjoint generating circles in M_z can be mapped onto the same generating circle in $M_{f(z)}$ (again, see Theorem D).

Proof (C1) Since $\text{End}(M) = \emptyset$, this follows from Lemma 8.

(C2) Since the argument is local (concerns only that part of the minimal set which projects onto U), we may simply assume that $E = B \times \Gamma$, $M_U = U \times \Gamma$ and to work with M_U rather than with \tilde{M}_U .

We have $(z, p) \in \text{Sint}_s(M)$. Consider an M -open neighborhood \mathcal{G} of (z, p) , mentioned in the definition of a strongly star-like interior point. One of the properties of \mathcal{G} is that if $b \in p(\mathcal{G})$ then \mathcal{G}_b contains the point (b, p) . Thus, it is sufficient to prove that $z \in p(\mathcal{G})$ is a limit point of $p(\mathcal{G})$. Suppose, on the contrary, that z is an isolated point of $p(\mathcal{G})$. Then $\mathcal{G} \cap M_z$ is an M -open neighborhood of (z, p) . Since $F|_M$ is minimal, (z, p) returns to $\mathcal{G} \cap M_z$ whence we obviously get that z is a periodic point of f . However, f is minimal and so B is just the periodic orbit of z under f , a contradiction with the infiniteness of B .

(C3) We claim that the set $\mathcal{R}_E(M)$ is closed (in E , hence also in M). To show this, let $\mathcal{R}_E(M) \ni \gamma_n \rightarrow \gamma \in E$. Since we work only with a neighborhood of the fibre containing γ , we may assume that $E = B \times \Gamma$. Denote $p(\gamma_n) = b_n$ and $p(\gamma) = b$. Since M is closed, $\gamma \in M$. However, $M = \text{Sint}_s(M)$ and so, by the definition of a star-like interior point, for large n the point γ_n has an M_{b_n} -open neighborhood whose second projection is a subset of the second projection of an M_b -open neighborhood of the point γ . Since $\gamma_n \in \mathcal{R}_E(M)$, this obviously implies that also $\gamma \in \mathcal{R}_E(M)$. We have thus proved that $\mathcal{R}_E(M)$ is closed, hence compact. Then also its projection $\mathcal{R}_B(M) = p(\mathcal{R}_E(M))$ is compact.

To prove that the (closed) set $\mathcal{R}_B(M)$ is nowhere dense, suppose, on the contrary, that some closed ball C is a subset of $\mathcal{R}_B(M)$ (closed balls here and in the rest of the proof of (C3) are always closed balls in the topology of B).

Combining Lemma 15(a) and Lemma 16 we get that there are a closed ball $C_1 \subseteq C$ and ramification points $q_1, q_2 \in \Gamma$ such that

$$C_1 \times \{q_1\} \subseteq \mathcal{R}_E(M) \quad \text{and} \quad F(C_1 \times \{q_1\}) = f(C_1) \times \{q_2\} \subseteq \mathcal{R}_E(M).$$

The set $f(C_1) \subseteq \mathcal{R}_B(M)$ has nonempty interior in B because C_1 has nonempty interior in B and $f : B \rightarrow B$, being a minimal map, is feebly open. Then, by Lemma 16(b), there is a closed ball C_2 and a ramification point q_3 of Γ such that

$$C_2 \subseteq f(C_1) \quad \text{and} \quad F(C_2 \times \{q_2\}) = f(C_2) \times \{q_3\} \subseteq \mathcal{R}_E(M).$$

Again, as above, $f(C_2)$ has nonempty interior in B and so we can apply Lemma 16 to find C_3 and q_4 . Continuing in this way, we obtain a sequence of closed balls $(C_n)_{n=1}^\infty$ in B and a sequence $(q_n)_{n=1}^\infty$ of ramification points of Γ such that

$$C_n \times \{q_n\} \subseteq \mathcal{R}_E(M) \quad \text{and} \quad F(C_n \times \{q_n\}) \supseteq C_{n+1} \times \{q_{n+1}\} \quad \text{for every } n.$$

Now choose a point γ in the nonempty compact set

$$(C_1 \times \{q_1\}) \cap F^{-1}(C_2 \times \{q_2\}) \cap F^{-2}(C_3 \times \{q_3\}) \cap \dots$$

Then all the points $\gamma, F(\gamma), F^2(\gamma), \dots$ belong to $\mathcal{R}_E(M)$. By minimality of $F|_M : M \rightarrow M$, the set $\mathcal{R}_E(M) \subseteq M$ containing the F -orbit of γ is dense in M . Since $\mathcal{R}_E(M)$ is also closed in M (see the beginning of the proof of (C3)), we get that $\mathcal{R}_E(M) = M$. However, this contradicts the fact that $\mathcal{R}_E(M)$ is nowhere dense in M . Indeed,

if $(z, g) \in \mathcal{R}_E(M)$ then it is a ramification point of M_z and a small connected Γ_z -neighborhood of (z, g) (which has the form of a star, a full sub-star of which is a subset of M_z) contains no other ramification points of Γ_z , while containing points from M_z different from (z, g) . It obviously follows that in every M -open neighborhood of (z, g) there is an M -ball disjoint with $\mathcal{R}_E(M)$.

(C4) Let H be the homeo-part of the minimal system (B, f) . Both the f -image and the f -pre-image of a nowhere dense set are nowhere dense (see Subsection 3.1). Therefore, since $\mathcal{R}_B(M)$ is nowhere dense in B by (C3), the set

$$H^* = H \setminus \bigcup_{n=-\infty}^{\infty} f^n(\mathcal{R}_B(M))$$

is residual, $f(H^*) = H^*$, every point of H^* has just one f -pre-image, and both $f|_{H^*}$ and $(f|_{H^*})^{-1}$ are minimal homeomorphisms. For any $w \in H^*$, the set M_w is a graph without end-points which, by definition of H^* , has no ramification point and so M_w is a circle or a disjoint union of several circles for all $w \in H^*$.

Suppose that, for some $a \in B$, the set M_a is not a union of circles. In our argument only E_U for a small neighborhood U of a will play a role, therefore we may assume that $E = B \times \Gamma$. So, $M_a = \{a\} \times M_a^\Gamma$ for some subgraph M_a^Γ of Γ . Choose $z_0 \in M_a^\Gamma$ such that $(a, z_0) \in M_a$ does not belong to any circle contained in M_a (it may belong to a circle in Γ_a). Then for all $b \in B$ sufficiently close to a , in the set M_b there is no circle containing the point (b, z_0) , since otherwise (due to closedness of M and the fact that there are only finitely many circles in Γ) also M_a would contain a circle containing (a, z_0) . Fix a point $y^* \in H^*$. Then its forward orbit under f is a subset of H^* and so, if we choose a point $z \in \Gamma$ with $(y^*, z) \in M_{y^*}$, for each $n = 0, 1, 2, \dots$ the point $F^n(y^*, z)$ belongs to one of the circles forming the set $M_{f^n(y^*)}$. It follows that the trajectory of (y^*, z) under F does not approach the point (a, z_0) , which contradicts the minimality of $F|_M$. Thus we have proved that all the sets M_b , $b \in B$, are unions of circles.

Now let m be the maximum number of (disjoint) circles in M_w for $w \in H^*$. Then $m \geq 1$. Fix a point $w \in H^*$ such that M_w consists of m circles. Since w has just one f -pre-image (and this pre-image belongs to H^*) and $F : M \rightarrow M$ is surjective, also $M_{f^{-1}(w)}$ consists of m disjoint circles (less than m circles cannot be continuously mapped onto m disjoint circles). By induction, $M_{f^{-j}(w)}$ consists of m disjoint circles for every $j = 0, 1, 2, \dots$. Since B is a compact metric space and $f : B \rightarrow B$ is minimal, the backward orbit $\{f^{-j}(w) : j = 0, 1, 2, \dots\}$ is dense in B (see Subsection 3.1). Since M is closed, the fact that M_w consists of m disjoint circles for every w in a dense subset of H^* implies (in view of the fact that there are only finitely many possibilities for a choice of m disjoint circles in Γ) that M_w consists of m disjoint circles for all $w \in H^*$ and M_w contains m disjoint circles (and perhaps some other circles) for all $w \in B \setminus H^*$. So, if we put

$$\mathcal{O} = \{z \in B : M_z \text{ is a disjoint union of } m \text{ circles}\},$$

then $B \setminus \mathcal{O} = \{z \in B : M_z \text{ is a union of circles properly containing } m \text{ disjoint circles}\}$.

Since $\mathcal{O} \supseteq H^*$, \mathcal{O} is dense in B . To prove that \mathcal{O} is open we are going to show that $B \setminus \mathcal{O}$ is closed. So, let $B \setminus \mathcal{O} \ni x_n \rightarrow x \in B$. Since we may assume that all the

points x_n are in a trivializing neighborhood of x , we may also assume that $E = B \times \Gamma$. Further, by passing to a subsequence if necessary, we may assume that for some disjoint circles C_1, \dots, C_m in Γ we have $M_{x_n} \supseteq \{x_n\} \times \bigcup_{i=1}^m C_i$ for every n . Taking into account that the points x_n belong to $B \setminus \mathcal{O}$ and again passing to a subsequence if necessary, we may assume that there is a circle S in Γ different from all C_i , $i = 1, \dots, m$, such that $M_{x_n} \supseteq \{x_n\} \times S$ for every n . Then, since M is closed, $M_x \supseteq \{x\} \times (S \cup \bigcup_{i=1}^m C_i)$ which implies that $x \in B \setminus \mathcal{O}$.

(C5) Let $z \in V \subseteq \mathcal{O}$ be a trivializing neighborhood. We may simply assume that $p^{-1}(V) = V \times \Gamma$. Then $M_z = \{z\} \times \bigcup_{i=1}^m C_i$ for some pairwise disjoint circles C_i in Γ . If a circle $C \subseteq \Gamma$ is different from these m circles, then M_v does not contain $\{v\} \times C$ whenever $v \in V$ is sufficiently close to z (otherwise the closed set M_z would contain $\{z\} \times C$). Thus, it is sufficient to choose a sufficiently small neighborhood $z \in U \subseteq V$.

From what we have just proved it follows that if $\mathcal{O} = B$ then M is a bundle with fibre equal to a disjoint union of m circles. Now additionally assume that $E = B \times \Gamma$ and B is connected. For every $x \in B$ the set M_x is a disjoint union of m circles (where m does not depend on $x \in B$). There are only finitely many m -tuples of circles in Γ and so, using the closedness of M and connectedness of B , we get that M is the product of B and some m -tuple of disjoint circles in Γ .

(C6) Since f is minimal, the f -pre-image of a residual set is residual and so there is a point $x \in \mathcal{O}$ whose forward orbit is a subset of \mathcal{O} . Choose a point in M_x . Since its forward orbit is dense in M and is a subset of $M_{\mathcal{O}}$, the result follows.

(C7) If $b \in \mathcal{O}$ then M_b is a disjoint union of m circles and each of them is generating by definition (even if the point b is isolated in B). Then (C4), (C6) show that every M_b is the union of generating circles (even if $b \in B \setminus \mathcal{O}$).

If $b \in B \setminus \mathcal{O}$, it is possible that each circle in M_b is generating as in Theorem D. However, it may contain also a non-generating circle. To see this, consider the case (12₃) in the proof of Theorem D. There, in one fibre of a minimal set, we can have two “geometric” circles having two points in common. This gives 6 circles altogether but only two of them, namely (in the notation from the proof of Theorem D) $\{c_l\} \times S_1$ and $\{c_l\} \times S_1^*$, are generating ones. However, at least m of the circles in M_b , $b \in B \setminus \mathcal{O}$ are disjoint generating circles. Indeed, consider a trivializing neighborhood W of b and think of E_W as being the product $W \times \Gamma$. Then just choose a sequence of points $b_n \in \mathcal{O}$, $b_n \rightarrow b$ such that every $M_{b_n} = \{b_n\} \times A$ for the union A of some fixed m disjoint circles in Γ (this is possible since Γ contains only finitely many combinations of disjoint m circles). Then $M_b \supseteq \{b\} \times A$ and so M_b contains at least m disjoint generating circles. Since $b \notin \mathcal{O}$, M_b cannot be just the union of these m circles and since we already know that M is a union of generating circles, M_b has to contain another generating circle.

(C8) Fix $z \in \mathcal{O}$. First we prove that if S is a circle in M_z then $F(S)$ is a circle in $M_{f(z)}$. We will work only with small neighborhoods of z and $f(z)$, therefore we may assume that $E = B \times \Gamma$. By (C5), we may fix a neighborhood $z \in U \subseteq \mathcal{O}$ such that

$$M_U = U \times \bigcup_{i=1}^m C_i \text{ where } C_1, \dots, C_m \text{ are pairwise disjoint circles in } \Gamma. \quad (6.1)$$

Set $S = \{z\} \times C$ where C is one of the circles C_i . We need to prove that $F(S) \subseteq M_{f(z)}$ is a circle.

Let us start by considering the case when $f(z) \in \mathcal{O}$. Then $M_{f(z)}$ is a disjoint union of circles and so $F(S)$ is necessarily a connected subset of one of them, call it T . To prove that $F(S) = T$ suppose, on the contrary, that $F(S)$ is a *proper* subset of the circle T . We are going to prove that then there exists a redundant open set for $F|_M$ (which will contradict the minimality of $F|_M$). If $F(S)$ is an arc in T , there are two non-overlapping arcs in S such that each of them is mapped onto $F(S)$. Hence there are also two disjoint arcs $\{z\} \times J_1$ and $\{z\} \times J_2$ in S such that $F(\{z\} \times J_1)$ is in the interior of $F(\{z\} \times J_2)$. Then (6.1) and the fact that the mentioned property of the point z carries over to all the points sufficiently close to z , easily imply the existence of a redundant open set for $F|_M$, as desired. It remains to check the case when $F(S)$ is only a singleton in T . Then the existence of a redundant open set for $F|_M$ is obvious if also for all v in a neighborhood of z we have that $F(\{v\} \times C)$ is a singleton. If such a neighborhood of z does not exist, then arbitrarily close to z there are points $v \in \mathcal{O}$ for which $F(\{v\} \times C)$ is not a singleton. By choosing such a point v close enough to z we can guarantee that $F(\{v\} \times C)$ is a *proper* subset of a circle, i.e. an arc. To find a redundant open set for $F|_M$, one can simply repeat the argument which was used above in the case when $F(S)$ was an arc. We have thus proved that $F(S) \subseteq M_{f(z)}$ is a circle if $f(z) \in \mathcal{O}$. It is a generating circle by definition, since $f(z) \in \mathcal{O}$.

Now consider the case when $f(z) \in B \setminus \mathcal{O}$. In $U \setminus \{z\}$ there is a sequence $z_n \rightarrow z$ such that $f(z_n) \in \mathcal{O}$ (otherwise some neighborhood of z would be mapped into $B \setminus \mathcal{O}$ which would contradict the fact that a minimal map sends open sets to sets with nonempty interior). Put $S_n = \{z_n\} \times C$ and $F(S_n) = \{f(z_n)\} \times K_n$, $n = 1, 2, \dots$. Then, by what we have proved above (note that both z_n and $f(z_n)$ are in \mathcal{O}), we know that $K_n \subseteq \Gamma$ is a circle for every n . However, there are only finitely many circles in Γ and so, by passing to a subsequence if necessary, we may assume that $K_n = K$ does not depend on n . Then obviously also $F(S) = \{f(z)\} \times K$ and so $F(S)$ is a circle, in fact a generating circle (because $f(z_n) \in \mathcal{O}$).

To finish the proof of (C8), it remains to show that different, hence disjoint, circles in M_z are mapped onto *disjoint* circles in $M_{f(z)}$.

Again, we start by considering a particular case when $f(z) \in \mathcal{O}$. By replacing U in (6.1) by a smaller neighborhood of z if necessary, we may assume, due to (C5), that $M_{f(U)} = f(U) \times \bigcup_{i=1}^m Q_i$ where Q_1, \dots, Q_m are pairwise disjoint circles in Γ . Let $S = \{z\} \times C$, $S' = \{z\} \times C'$ be disjoint circles in M_z (here $C, C' \in \{C_1, \dots, C_m\}$, see (6.1)). To prove that also the circles $F(S)$ and $F(S')$ are disjoint, suppose on the contrary that $F(S) = F(S') = \{f(z)\} \times Q$ for some $Q \in \{Q_1, \dots, Q_m\}$. The circle $\{f(z)\} \times Q$ has positive distance from the rest of $M_{f(z)}$. Therefore, in view of (6.1), for all v sufficiently close to z it holds that both $\{v\} \times C$ and $\{v\} \times C'$ are mapped by F onto the same circle $\{f(v)\} \times Q$. The existence of a redundant open set for $F|_M$ easily follows; a contradiction.

Finally, consider the case when $f(z) \in B \setminus \mathcal{O}$. Again, let $S = \{z\} \times C$, $S' = \{z\} \times C'$ be disjoint circles in M_z . Choose a sequence $U \setminus \{z\} \ni z_n \rightarrow z$ such that $f(z_n) \in \mathcal{O}$. Consider the circles $S_n = \{z_n\} \times C$ and $S'_n = \{z_n\} \times C'$. For each n , both z_n and $f(z_n)$ are in \mathcal{O} and therefore, as we already know, $F(S_n) = \{f(z_n)\} \times P_n$ and $F(S'_n) = \{f(z_n)\} \times P'_n$ are disjoint circles. By passing to a subsequence if necessary, we may

assume that $P_n = P$ and $P'_n = P'$ do not depend on n . Then obviously $F(S) = \{f(z)\} \times P$ and $F(S') = \{f(z')\} \times P'$ which means that $F(S)$ and $F(S')$ are disjoint circles.

(C9) Let $S \subseteq M_z$ be a generating circle. So, there are circles $S_n \subseteq M_{z_n}$, $z_n \in \mathcal{O}$ (hence $z_n \neq z$), $n = 1, 2, \dots$, such that $S_n \rightarrow S$ with respect to the Hausdorff metric. By (C8), $F(S_n)$ is a generating circle for every n . Since $F(S_n) \rightarrow F(S)$ in the Hausdorff metric, $F(S)$ is a generating circle. Now see the proof of Theorem D, the case (12₃). The set $M_{c_l}^*$ consists of two circles, one “inside” the other. Together there are six circles there, two generating and four non-degenerating. Straightforward analysis shows that images of two non-degenerating circles are just arcs, not circles.

(C10) Let f be monotone. Suppose that $B \setminus \mathcal{O} \neq \emptyset$. To show that this leads to a contradiction, consider two cases.

If for every $z \in B \setminus \mathcal{O}$ the set $f^{-1}(z)$ intersects $B \setminus \mathcal{O}$, then there is a backward orbit of f lying in $B \setminus \mathcal{O}$. However, $B \setminus \mathcal{O}$ is nowhere dense while every backward orbit of a minimal map is dense, a contradiction.

If there exists $z_0 \in B \setminus \mathcal{O}$ such that the connected set $f^{-1}(z_0)$ is a subset of \mathcal{O} , we get a contradiction as follows. Fix a point $a \in f^{-1}(z_0)$. Since now we are going to find a special neighborhood of a by considering just small neighborhoods of a and z_0 , we may assume for a moment that $E = B \times \Gamma$. By (C5), there is a small neighborhood U_a of a such that $U_a \subseteq \mathcal{O}$ and $M_{U_a} = U_a \times \bigcup_{i=1}^m C_i^a$ where C_1^a, \dots, C_m^a are pairwise disjoint circles in Γ . By (C8), $F(\{a\} \times C_i^a) = \{f(a)\} \times K_i^a$, $i = 1, \dots, m$, for some pairwise disjoint circles K_1^a, \dots, K_m^a in Γ . Since there are only finitely many circles in Γ , there is $\varepsilon_0 > 0$ such that any two different (not necessarily disjoint) circles in Γ have Hausdorff distance at least ε_0 . Therefore, if $i \in \{1, \dots, m\}$ and if $u \in U_a$ is sufficiently close to a then the set $F(\{u\} \times C_i^a)$, which is a circle by (C8), equals $\{f(u)\} \times K_i^a$. By replacing U_a by a smaller neighborhood if necessary, we may assume that the last claim works for all $u \in U_a$. Finally, consider the relative neighborhood of a in $f^{-1}(z_0)$ of the form $V_a = U_a \cap f^{-1}(z_0)$. Denote also $S_i^a = \{z_0\} \times K_i^a$. Then we have that

$$\text{for every } v \in V_a, \quad F(M_v) = \bigcup_{i=1}^m S_i^a \subseteq M_{z_0}. \quad (6.2)$$

Without our above temporary assumption that $E = B \times \Gamma$, of course still a small relative neighborhood V_a of a exists such that (6.2) works for some pairwise disjoint circles S_1^a, \dots, S_m^a in M_{z_0} . Remember that, given $a \in f^{-1}(z_0)$, the family of these circles does not depend on the choice of $v \in V_a$.

Let $V_{a(1)}, \dots, V_{a(r)}$ be a finite cover of the compact space $f^{-1}(z_0)$ (in the relative topology), chosen from the open cover $\{V_a : a \in f^{-1}(z_0)\}$. Then, since $F(M) = M$, we have, by (6.2),

$$M_{z_0} = \bigcup_{a \in f^{-1}(z_0)} F(M_a) = \bigcup_{j=1}^r F(M_{V_{a(j)}}) = \bigcup_{j=1}^r \bigcup_{i=1}^m S_i^{a(j)}. \quad (6.3)$$

We claim that the family of m disjoint circles $\{S_1^{a(j)}, \dots, S_m^{a(j)}\}$ does not depend on j . To see it, fix $j, k \in \{1, \dots, r\}$, $j \neq k$. In particular case when $V_{a(j)} \cap V_{a(k)} \neq \emptyset$ it suffices

to choose $x \in V_{a(j)} \cap V_{a(k)}$ and to use that, by (6.2), it holds $\bigcup_{i=1}^m S_i^{a(j)} = F(M_x) = \bigcup_{i=1}^m S_i^{a(k)}$. In general case realize that in the family $V_{a(1)}, \dots, V_{a(r)}$ there is a finite chain of sets starting with $V_{a(j)}$ and ending with $V_{a(k)}$ such that any two consecutive elements of the chain intersect (if such a chain did not exist, the *connected* set $f^{-1}(z_0)$ would be a union of two *disjoint* nonempty sets open in the topology of $f^{-1}(z_0)$). Hence also in the general case we have $\bigcup_{i=1}^m S_i^{a(j)} = \bigcup_{i=1}^m S_i^{a(k)}$. Then (6.3) implies that M_{z_0} is a union of just m disjoint circles. Hence $z_0 \in \mathcal{O}$, a contradiction.

(C11) We claim that to prove $\mathcal{O} = B$ we may without loss of generality assume that B is also connected. In fact, suppose for a moment that we have proved $\mathcal{O} = B$ under the additional assumption of connectedness of B . Then we can finish the proof as follows. The space B , being compact and locally connected, has finitely many components B_1, \dots, B_r and these are locally connected. The map f , being minimal, cyclically permutes them and f^r is minimal on each of them. Then, for $i = 1, \dots, r$, the set M_{B_i} is a minimal set of $F^r|_{B_i \times \Gamma}$. Hence, using our temporary assumption that \mathcal{O} is the whole base space provided the base space is locally connected and connected, we get that for every $x \in B_i$ the set M_x is a disjoint union of m_i circles (where m_i does not depend on $x \in B_i$). There are only finitely many m_i -tuples of circles in Γ and so, using the closedness of M_{B_i} and connectedness of B_i , we get that M_{B_i} is the product of B_i and some m_i -tuple of disjoint circles in Γ . Further, by (C4) the positive integer m_i does not depend on i , i.e. there is m with $m_i = m$ for all $i = 1, \dots, r$. Thus, $\mathcal{O} = B$ (still, if $r > 1$, the m_i -tuple of circles may depend on i).

So, assume that the locally connected space B is also *connected*. We are going to prove that then $\mathcal{O} = B$.

Consider the open set $\mathcal{O} \subseteq B$ defined in (C4). Since B is locally connected, so is \mathcal{O} . Recall that a space X is locally connected if and only if for every open set U of X , each component of U is open. It follows that the set \mathcal{O} can be represented as a disjoint union $\mathcal{O} = \bigcup \mathcal{W}_j$ of a countable family of its components \mathcal{W}_j , each \mathcal{W}_j being B -open, locally connected and connected. Let m be the positive integer from (C4).

Due to the connectedness of $\mathcal{W}_j \subseteq \mathcal{O}$, by (C5) we obtain the direct product structure of each $M_{\mathcal{W}_j}$, i.e., there exist *pairwise disjoint* circles C_1^j, \dots, C_m^j in Γ such that

$$M_{\mathcal{W}_j} = \mathcal{W}_j \times \bigcup_{i=1}^m C_i^j. \quad (6.4)$$

The circles C_1^j, \dots, C_m^j in general depend on j , but m does not. Let L be the (finite) set of all circles C_i^j (for all j and all $i = 1, \dots, m$).

Since M is a closed set, for the closure of $M_{\mathcal{W}_j}$ we have $\overline{M_{\mathcal{W}_j}} = \overline{\mathcal{W}_j} \times \bigcup_{i=1}^m C_i^j \subseteq M$. The set $\overline{\mathcal{W}_j}$ is connected. We call each of m connected components $\overline{\mathcal{W}_j} \times C_i^j$ of the closure $\overline{M_{\mathcal{W}_j}}$ a *prime cylinder* (more precisely, $\overline{\mathcal{W}_j} \times C_i^j$ is a prime cylinder corresponding to the circle C_i^j). Each prime cylinder has nonempty E -interior. Notice also that each prime cylinder is a union of generating circles and is of course a connected subset of M . For a fixed circle C in Γ , consider the set of indices $I(C) := \{j : \mathcal{W}_j \times C \subseteq M\}$.

Let $[C_\alpha]$ be the components of the set $\overline{\bigcup_{j \in I(C)} \mathcal{W}_j} \times C$, so

$$\overline{\bigcup_{j \in I(C)} \mathcal{W}_j} \times C = \bigsqcup_{\alpha} [C_\alpha]. \quad (6.5)$$

We will say that each $[C_\alpha]$ is a *maximal cylinder* corresponding to the circle C (note that it is a subset of M). Observe that $[C_\alpha]$ has the form

$$[C_\alpha] = \overline{\bigcup_{\gamma \in \mathcal{Y}} P_\gamma} \text{ where } P_\gamma, \gamma \in \mathcal{Y} \text{ are some prime cylinders corresponding to } C. \quad (6.6)$$

By definition, $[C_\alpha] \cap [C_\beta] = \emptyset$ for $\alpha \neq \beta$. We will also need the following claim.

Claim (Properties of maximal cylinders).

- (a) Two maximal cylinders $\mathcal{M}_1, \mathcal{M}_2$ corresponding to the same circle C either are disjoint or coincide.
- (b) If $b \in \mathcal{O}$ and $\mathcal{M}_1 \cap \mathcal{M}_2 \cap M_b \neq \emptyset$, then $\mathcal{M}_1 = \mathcal{M}_2$.
- (c) If \mathcal{M} and $\mathcal{M}_\lambda, \lambda \in \Lambda$ are maximal cylinders with $M \subseteq \bigcup_{\lambda \in \Lambda} M_\lambda$, then $\mathcal{M} = \mathcal{M}_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.
- (d) For any $k \geq 1$, the F^k -image of a prime cylinder $P = \overline{\mathcal{W}_i} \times C \subseteq M$ is a subset of a maximal cylinder.
- (e) The family of all maximal cylinders is finite and its union equals M .
- (f) Any maximal cylinder is mapped by F into a maximal cylinder.

Proof of Claim (a) Each of the sets $\mathcal{M}_1, \mathcal{M}_2$ is a component of the set $\overline{\bigcup_{j \in I(C)} \mathcal{W}_j} \times C$. Two non-disjoint components coincide.

(b) By (C4), M_b is a *disjoint* union of circles. One of them, call it C , is such that $\mathcal{M}_1 \cap \mathcal{M}_2$ intersects $\{b\} \times C$ and, by definition of maximal cylinders, $(\mathcal{M}_1)_b = (\mathcal{M}_2)_b = \{b\} \times C$. Now apply (a).

(c) By definition of a maximal cylinder, there exists $b \in \mathcal{O}$ and $C \in L$ such that $\mathcal{M} \supseteq \{b\} \times C$. There is $\lambda_0 \in \Lambda$ such that \mathcal{M}_{λ_0} intersects $\{b\} \times C$. By (b), $\mathcal{M} = \mathcal{M}_{\lambda_0}$.

(d) The set P is a union of generating circles and, by (C8) and (C9), a generating circle is mapped onto a (generating) circle. It follows that if S is a circle in Γ then, due to continuity of F^k and the fact that Γ contains only finitely many circles, the set of those $z \in \overline{\mathcal{W}_i}$ for which $F^k(\{z\} \times C) = \{f^k(z)\} \times S$, is open in $\overline{\mathcal{W}_i}$. However, the set $\overline{\mathcal{W}_i}$ is connected. Therefore there exists one circle S such that

$$F^k(P) = f^k(\overline{\mathcal{W}_i}) \times S \subseteq M. \quad (6.7)$$

Since $f : B \rightarrow B$ is minimal, it is feebly open. Hence f^k is feebly open. Therefore the set $U_i := \text{Int } f^k(\mathcal{W}_i)$ of all B -interior points of $f^k(\mathcal{W}_i)$ is dense in $f^k(\mathcal{W}_i)$, hence also dense in $\overline{f^k(\mathcal{W}_i)} = f^k(\overline{\mathcal{W}_i})$. So, $\overline{U_i} = f^k(\overline{\mathcal{W}_i})$. On the other hand, U_i is open and $\mathcal{O} = \bigcup_j \mathcal{W}_j$ is dense and open, hence

$$f^k(\overline{\mathcal{W}_i}) = \overline{U_i} = \overline{\mathcal{O} \cap U_i} = \overline{(\bigcup_j \mathcal{W}_j) \cap U_i} = \overline{\bigcup_j (\mathcal{W}_j \cap U_i)}. \quad (6.8)$$

Further, by (6.7), $U_i \times S \subseteq M$. So, if $\mathcal{W}_l \cap U_i \neq \emptyset$ for some l , then $\mathcal{W}_l \times S \subseteq M$, i.e. $l \in I(S)$. It follows, taking into account (6.7) and (6.8), that

$$F^k(P) = f^k(\overline{\mathcal{W}_i}) \times S = \overline{\bigcup_{l \in I(S)} (\mathcal{W}_l \cap U_i)} \times S \subseteq \overline{\bigcup_{l \in I(S)} \mathcal{W}_l} \times S = \bigsqcup_{\alpha} [S_\alpha]$$

where $[S_\alpha]$ are the components of the set $\overline{\bigcup_{I \in I(S)} \mathcal{W}_I} \times S$ (see (6.5)). Since $F^k(P)$ is connected, it is a subset of one S_α , which finishes the proof that F^k -image of a prime cylinder is a subset of a maximal cylinder.

(e) Now, since the prime cylinder P has nonempty interior in M and $F|_M$ is minimal, we have that $M = \bigcup_{k=0}^{N-1} F^k(P)$ for some N (this is a property of minimal systems, see Subsection 3.1). This together with (d) give that M is covered by N (not necessarily distinct) maximal cylinders. Then, using (c), we get that the family of all maximal cylinders is finite (has at most N elements) and its union equals M .

(f) Let $\mathcal{M}_1, \dots, \mathcal{M}_r$ be the list of all (pairwise distinct) maximal cylinders (at the moment we do not know whether they are pairwise disjoint). For $i = 1, \dots, r$ put $\mathcal{M}_i = B_i \times S_i$, where $B_i \subseteq B$ is closed and connected set (containing at least one of the sets \mathcal{W}_j) and S_i is a circle in Γ (in fact $S_i \in L$, see (6.4) and the notation L after it). We prove that, for instance, $F(\mathcal{M}_1)$ is a subset of a maximal cylinder. By (6.6),

$$\mathcal{M}_1 = \overline{\bigcup_{\gamma \in Y} P_\gamma} \quad \text{where } P_\gamma, \gamma \in Y \text{ is the family of prime cylinders contained in } \mathcal{M}_1$$

(of course, all these prime cylinders P_γ correspond to the circle S_1). We know, by (d), that for each $\gamma \in Y$ there is a maximal cylinder, call it \mathcal{N}_γ , with $F(P_\gamma) \subseteq \mathcal{N}_\gamma$. In the particular case when all these maximal cylinders are the same, i.e. when there is $\gamma_0 \in Y$ such that $\mathcal{N}_\gamma = \mathcal{N}_{\gamma_0}$ for all $\gamma \in Y$, we get the desired relation:

$$F(\mathcal{M}_1) = F(\overline{\bigcup_{\gamma \in Y} P_\gamma}) = \overline{F(\bigcup_{\gamma \in Y} P_\gamma)} = \overline{\bigcup_{\gamma \in Y} F(P_\gamma)} \subseteq \mathcal{N}_{\gamma_0}.$$

To finish the proof, we are going to show that the assumption that not all maximal cylinders \mathcal{N}_γ are the same, leads to a contradiction.

So, let $d \geq 2$ and $\mathcal{N}^1, \dots, \mathcal{N}^d$ be the list of all pairwise distinct maximal cylinders in the family \mathcal{N}_γ , $\gamma \in Y$. Then there is a decomposition $Y = Y^1 \sqcup \dots \sqcup Y^d$ such that $F(P_\gamma) \subseteq \mathcal{N}^j$ for all $\gamma \in Y^j$. Denote $\Pi_j := \overline{\bigcup_{\gamma \in Y^j} P_\gamma}$, $j = 1, 2, \dots, d$. Of course,

$$F(\Pi_j) = F(\overline{\bigcup_{\gamma \in Y^j} P_\gamma}) \subseteq \mathcal{N}^j.$$

We claim that the sets Π_j are pairwise disjoint. To show this, suppose on the contrary that $\Pi_i \cap \Pi_k \neq \emptyset$ for some $i \neq k$. Then, in view of the fact that all prime cylinders P_γ correspond to the circle S_1 , there is $b_0 \in B$ such that $\{b_0\} \times S_1 \subseteq \Pi_i \cap \Pi_k$. Obviously, $\{b_0\} \times S_1$ is a generating circle and, by (C8) and (C9), its F -image is some circle $\{f(b_0)\} \times S^*$. Then $\mathcal{N}^i = B^i \times S^*$ and $\mathcal{N}^k = B^k \times S^*$ for some closed sets B^i, B^k containing $f(b_0)$. Here $\mathcal{N}^i, \mathcal{N}^k$ are different, but not disjoint, maximal cylinders corresponding to the same circle S^* . This contradicts already proved part (a) of the Claim. So, we have proved that the sets Π_j are pairwise disjoint. Then

$$\mathcal{M}_1 = \overline{\bigcup_{\gamma \in Y} P_\gamma} = \bigcup_{j=1}^d \overline{\bigcup_{\gamma \in Y^j} P_\gamma} = \bigcup_{j=1}^d \Pi_j$$

is the decomposition of the connected set \mathcal{M}_1 into finitely many closed nonempty sets, a contradiction. This finishes the proof of Claim. $\checkmark \checkmark \checkmark$

Now we are ready to finish the proof of (C11).

Similarly as in the proof of the part (f) of Claim, let $\mathcal{M}_1, \dots, \mathcal{M}_r$ be the list of all (pairwise distinct) maximal cylinders, where $\mathcal{M}_i = B_i \times S_i$. By the part (b) of Claim,

two different maximal cylinders may intersect only in fibres above the set $B \setminus \mathcal{O}$. Therefore

$$\mathcal{M}_1 \setminus (\mathcal{M}_2 \cup \dots \cup \mathcal{M}_r) \quad \text{has nonempty interior in } M. \quad (6.9)$$

Since the map $F|_M$ is minimal, there exists a positive integer j with

$$F^j(\mathcal{M}_1) \cap (\mathcal{M}_1 \setminus (\mathcal{M}_2 \cup \dots \cup \mathcal{M}_r)) \neq \emptyset.$$

However, every maximal cylinder is mapped by F into a maximal cylinder, therefore we necessarily have $F^j(\mathcal{M}_1) \subseteq \mathcal{M}_1$. It follows that $F^j|_{\mathcal{M}_1} : \mathcal{M}_1 \rightarrow \mathcal{M}_1$ is minimal. (Indeed, if in a minimal system (M, F) there is a closed and connected set $Y \neq \emptyset$ with $F^j(Y) \subseteq Y$ for some $j \geq 1$, then Y is a minimal set of F^j . This is probably well known and explicitly can be found in [28].)

However, f is minimal on the *connected* space B (see the discussion at the beginning of the proof of (C11)), hence it is totally minimal (this is well known, see e.g. [28]). Since the minimal map f^j is the base map of F^j , the fact that $F^j|_{\mathcal{M}_1}$ is minimal implies that $B_1 = B$. In the same way we get $B_i = B$ for all $i = 1, \dots, r$. But then M is a finite union (see Claim (e)) of maximal cylinders in the form $\mathcal{M}_i = B \times S_i$, $i = 1, \dots, r$. Since the maximal cylinders \mathcal{M}_i of this particular form are assumed to be pairwise distinct and $\mathcal{O} \neq \emptyset$, by Claim (b) they are pairwise disjoint (i.e., the circles S_i are pairwise disjoint). Thus $\mathcal{O} = B$. The sets \mathcal{M}_i are the components of the minimal set M and so they are cyclically permuted by F . \square

7 Proof of Theorem E

Lemma 17 *Let M be a nowhere dense closed subset of a compact graph bundle (E, B, p, Γ) . Then a typical fibre of M is totally disconnected.*

Proof Suppose, on the contrary, that $A = \{b \in B : M_b \text{ is not totally disconnected}\}$ is of 2nd category in B . Of course, M_b is not totally disconnected if and only if it contains an arc and since Γ is a graph, this is if and only if M_b contains a ball in Γ_b . Therefore, since A is of 2nd category, there is $n_0 \in \mathbb{N}$ such that

$$A_{n_0} = \{b \in B : M_b \text{ contains a ball in } \Gamma_b \text{ with radius } \geq 1/n_0\}$$

is of 2nd category. Since B is covered by finitely many trivializing neighborhoods, there is a trivializing neighborhood U such that $A_{n_0} \cap U$ is of 2nd category. To get a desired contradiction, it is sufficient to show that $M \cap p^{-1}(U)$ is somewhere dense. Of course, we may without loss of generality assume that $p^{-1}(U) = U \times \Gamma$. To prove that $M \cap (U \times \Gamma)$ is somewhere dense, fix a countable dense set $S \subseteq \Gamma$. For $b \in U$ and $s \in S$, a ball in $\Gamma_b = \{b\} \times \Gamma$ whose radius is $\geq 1/n_0$ and whose center has distance from $\{b\} \times \{s\}$ less than $1/(2n_0)$ is in the sequel said to be a *big ball centered close to level s* . Let

$$A_{n_0}^{U,s} = \{b \in U : M_b \text{ contains a big ball centered close to level } s\}.$$

It is obvious that $A_{n_0} \cap U = \bigcup_{s \in S} A_{n_0}^{U,s}$ and so there is $s_0 \in S$ such that $A_{n_0}^{U,s_0}$ is of 2nd category, hence dense in some nonempty open set $G_1 \subseteq U$. On the other hand,

any ball in Γ whose radius is $\geq 1/n_0$ and whose center has distance from s_0 less than $1/(2n_0)$, contains the ball G_2 with center s_0 and radius $1/(2n_0)$. Then M , being closed, contains the open set $G_1 \times G_2$. This contradicts nowhere density of M . \square

Proposition 3 *In Theorem A, suppose that $\text{card} M_z < \infty$ for some z in the homeo-part H of f . Then a typical fibre of the minimal set M has cardinality $N := \min\{\text{card} M_x : x \in H\} < \infty$.*

Proof By the assumption, $N \leq \text{card} M_z$ is a positive integer and there is $z_0 \in H$ with $\text{card} M_{z_0} = N$. Denote

$$B^{(\leq N)} := \{x \in B : \text{card} M_x \leq N\}.$$

Then $z_0 \in B^{(\leq N)}$ and we claim that $B^{(\leq N)} = \bigcap_{n=1}^{\infty} G_{1/n}^{(N)}$ where $G_{1/n}^{(N)}$ is the set of those points $x \in B$ for which M_x can be covered by a disjoint union of N open sets in the fibre $p^{-1}(x)$, each of these sets having diameter $< 1/n$. The inclusion $B^{(\leq N)} \subseteq \bigcap_{n=1}^{\infty} G_{1/n}^{(N)}$ is trivial. To prove the converse inclusion, realize that simultaneous assumptions $x \in \bigcap_{n=1}^{\infty} G_{1/n}^{(N)}$ and $\text{card} M_x \geq N+1$ obviously give a contradiction. Since M is compact, $G_{1/n}^{(N)}$ is open. So, $B^{(\leq N)}$ is a G_δ set in B . Moreover, we claim that it is dense in B . To see this, realize that the set H is f -invariant and for $x \in H$ we have $\text{card} M_x \geq \text{card} M_{f(x)} \geq N$. Hence, since for $z_0 \in H$ we have $\text{card} M_{z_0} = N$, we get that $\text{card} M_{f^k(z_0)} = N$ for all $k = 0, 1, \dots$. So the set $B^{(\leq N)}$ contains the whole (forward) orbit of z_0 , which is dense by minimality of f . We have proved that $B^{(\leq N)}$ is a G_δ dense set in B .

For each x in the G_δ dense set $H \cap B^{(\leq N)}$ it obviously holds $\text{card} M_x = N$. \square

Theorem E. *Let M be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle (E, B, p, Γ) . Assume that M is nowhere dense. Then either*

- (E1) *a typical fibre of M is a Cantor set, or*
- (E2) *there is a positive integer N such that a typical fibre of M has cardinality N .*

Proof Below, we use some ideas from the proof of [21, Theorem 6.1].

By Lemma 17, a typical fibre of M is totally disconnected. If (E1) does not hold,

$$B^{\text{isol}} := \{b \in B : M_b \text{ has an isolated point}\}$$

is a 2nd category set. Since B can be covered by a finite number of trivializing neighborhoods, one of them, call it W , is such that $B^{\text{isol}} \cap W$ is of 2nd category in W (even, of 2nd category in B).

We fix a homeomorphism $h : p^{-1}(W) \rightarrow W \times \Gamma$ such that on $p^{-1}(W)$ it holds $\text{pr}_1 \circ h = p$.

Let \mathcal{T} be a countable family of subtrees of the fibre Γ such that the interiors, in the topology of Γ , of them are connected (i.e., the interior of such a tree is obtained from the tree by possible removing of some or all of the endpoints of the tree; no

point which is not an endpoint is removed) and these interiors form a base of the topology on Γ . Consider the countable set

$$\mathcal{D} := \{(T_1^\Gamma, T_2^\Gamma) : T_1^\Gamma, T_2^\Gamma \in \mathcal{T} \text{ and } T_1^\Gamma \subseteq \text{Int} T_2^\Gamma\}.$$

Note that the homeomorphism h induces corresponding families of trees in each fibre $p^{-1}(b)$, $b \in W$. For each pair $(T_1^\Gamma, T_2^\Gamma) \in \mathcal{D}$, put

$$W(T_1^\Gamma, T_2^\Gamma) := \{b \in W : M_b \cap T_{2,b} = M_b \cap T_{1,b} \text{ is a singleton}\} \quad (7.1)$$

where $T_{i,b} := h^{-1}(\{b\} \times T_i^\Gamma)$ is the tree in $p^{-1}(b)$ corresponding to T_i^Γ , $i = 1, 2$. Of course,

$$B^{\text{isol}} \cap W = \bigcup_{(T_1^\Gamma, T_2^\Gamma) \in \mathcal{D}} W(T_1^\Gamma, T_2^\Gamma).$$

Since $B^{\text{isol}} \cap W$ is a 2nd category set, there is a pair $(\tilde{T}_1^\Gamma, \tilde{T}_2^\Gamma) \in \mathcal{D}$ such that $W(\tilde{T}_1^\Gamma, \tilde{T}_2^\Gamma)$ is dense in an open set $U \subseteq W$.

Let $\mathcal{K}(E)$ be the (compact) space of all compact subsets of E endowed with the Hausdorff distance generated by the original metric in E . Since M is compact, the map $\Theta : B \rightarrow \mathcal{K}(E)$ defined by $\Theta(b) = M_b$, $b \in B$ is upper semicontinuous. Hence, see e.g. [1, Theorem 1.4.13], the set $C(\Theta)$ of continuity points of Θ is residual in B . By Lemma 5, there is an invariant residual set R in B such that $R \subseteq C(\Theta) \cap H$ where H is the homeo-part of f .

Denote $V := \text{Int} \tilde{T}_2^\Gamma$. We claim that for any $b \in U \cap R \subseteq \overline{W(\tilde{T}_1^\Gamma, \tilde{T}_2^\Gamma)} \cap R$ it holds that $M_b^\Gamma \cap V$ is a singleton. In fact, each such point b is a limit of points from $W(\tilde{T}_1^\Gamma, \tilde{T}_2^\Gamma)$ and so $M_b^\Gamma \cap \tilde{T}_1^\Gamma$ contains a point. Suppose that $M_b^\Gamma \cap V$ contains more than one point. Then, since b is a point of continuity of Θ , also for those points $c \in W(\tilde{T}_1^\Gamma, \tilde{T}_2^\Gamma) \cap U$ which are sufficiently close to b , we get that $M_c^\Gamma \cap V$ contains at least two points, which contradicts (7.1).

The set $\mathcal{O} := M \cap h^{-1}(U \times V)$ is a nonempty open subset of M . Hence, by the well known property of compact minimal systems, there is a positive integer n_0 such that every point from M visits \mathcal{O} not later than after n_0 iterations.

Now fix $y \in R$ and $e \in M_y$. By what was said above, $F^{n(e)}(e) \in \mathcal{O}$ for some $n(e) \leq n_0$. Hence $F^{n(e)}(G(e)) \subseteq \mathcal{O}$ for some neighborhood $G(e)$ of e in M_y . It follows that

$$F^{n(e)}(G(e)) \subseteq h^{-1}(\{f^{n(e)}(y)\} \times V) \cap M \subseteq \mathcal{O}. \quad (7.2)$$

By definition of \mathcal{O} and the fact that y has been chosen in the invariant set R , we get that $f^{n(e)}(y) \in U \cap R$. Therefore, by (7.2), $F^{n(e)}(G(e))$ is a singleton. Then also $F^{n_0}(G(e))$ is a singleton. Since M_y is compact, there are finitely many points $e_1, \dots, e_k \in M_y$ such that $G(e_1) \cup \dots \cup G(e_k) = M_y$. It follows that $F^{n_0}(M_y)$ is a finite set (it is a subset of $M_{f^{n_0}(y)}$ with cardinality $\leq k$). Since $y \in H$ and $F(M) = M$, also $f^{n_0}(y) \in H$ and $M_{f^{n_0}(y)} = F^{n_0}(M_y)$ is finite. So, one can apply Proposition D to get (E2).

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